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FIXED AND PERIODIC POINTS UNDER SET-VALUED MAPPINGS

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In this paper we prove some theorems on fixed and periodic points of set-valued mappings in a metric space which are generalisations of some of the works of Edelstein [*J. Lond. Math. Soc.* **37** (1962), 74–79] and Ćirić [*Fund. Math.* **120** (1984), 223–28] on point-valued mappings.

1. INTRODUCTION

Let X be a metric space with metric d and $B(X)$ be the family of all non-empty bounded subsets of X. A set-valued mapping $F : X \to B(X)$ is said to have a fixed point $z \in X$ if $z \in Fz$ {(Nadler[14], Kaulgud and Pai[13]}). Using this definition, Kaulgud and Pai[13] have generalized a number of theorems of Kannan[10-12], on point-valued mappings. Pursuing with this idea, Fisher[7-9] proved some results on fixed point and common fixed point of set-valued mappings which are generalizations of the results of Ćirić[2] and Fisher[6].

If $T : X \to X$ is a point-valued mapping, Edelstein[5] proved a fixed point theorem for a contractive operator and a periodic point theorem for a ϵ-contractive operator, the latter result being recently generalised by Ćirić[3] using the concept of well-locally contractive operators. Ćirić[3] also showed that in addition to well locally contractiveness if one assumes certain other hypothesis, then a periodic point turned out to be a fixed point. In this paper we attempt to generalize these theorems to set-valued mappings. It may be noted that the concept of (ϵ, λ) uniformly local contractiveness of Edelstein[4] has been generalised to set-valued mappings by Nadler[14].

If $A, B \in B(X)$ then to investigate the existence of fixed point for set-valued mappings, some authors used Hausdorff metric $D(A, B)$ (Nadler[14], Kaulgud and Pai [3]) while others used $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$ (Fisher[7,9]). Here we adhere to the Hausdorff metric $D(A, B)$ because use of the distance $\delta(A, B)$ becomes ineffective in our cases as it appeared to be applicable only to isolated sets, a very special

case. A mapping $f: X \to X$ is called contractive[5] if $d[f(p), f(q)] < d(p, q)$ for all $p, q \in X$, $p \neq q$ and ϵ-contractive[5] if there exists $\epsilon > 0$ such that $d[f(p), f(q)] < d(p, q)$ for all $p, q \in X$, $p \neq q$ and whenever $d(p, q) < \epsilon$.

A mapping $f: X \to X$ is called well locally contractive[3] if for each $u \in X$ there exists a sphere $S(u, r(u))$ such that $x, y \in S(u, r(u))$ with $x \neq y$ implies

$$d(Tx, Ty) < d(x, y) \text{ and } Tx, Ty \in S(v, r(v))$$

for some $v \in X$.

Let $\{A_n\}$, $n = 1, 2, 3, \ldots$ be a sequence of non-empty subsets of X. The sequence $\{A_n\}$ is said to converge[8] to the subset A of X if (i) each point a in A is the limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$ for $n = 1, 2, 3, \ldots$ (ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A(\epsilon)$ for $n > N$, where $A(\epsilon)$ denotes the set of all points x in X for which there exists a point a in A, depending on X, such that $d(x, a) < \epsilon$. The set A is then said to be the limit of the sequence $\{A_n\}$.

The mapping $F: X \to B(X)$ is said to be continuous[8] at $x \in X$ if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is a continuous mapping of X into $B(x)$ if it is continuous at each point x in X. The symbol Cpt (X) wi!l stand for the family of all non-empty compact subsets of X.

2. Lemmas

Lemma 1—Let $F: X \to B(X)$ be such that $D(Fx, Fy) < d(x, y)$, $x, y \in X$, $x \neq y$. Then F is continuous.

Proof : Let $x_n \to x$. We may clearly suppose that $x_n \neq x$ for all large n. Let $\epsilon > 0$ be arbitrary. There exists a positive integer N such that

$$D(Fx_n, Fx) < d(x_n, x) < \epsilon \text{ for } n > N.$$

So, $Fx_n \subset Fx(\epsilon)$ and also $Fx \subset Fx_n(\epsilon)$ for $n > N$ and this shows that $Fx_n \to Fx$ as $n \to \infty$ and the proof is complete.

Lemma 2—If $\{A_n\}$ and $\{B_n\}$ are sequences from $B(X)$ which converge respectively to the sets A and $B \in B(X)$ then the sequence $\{D(A_n, B_n)\}$ converges to $D(A, B)$.

Proof : Let $\epsilon > 0$ be arbitrary. There exists a positive integer N such that

$$D(A_n, A) < \epsilon/2 \text{ and } D(B_n, B) < \epsilon/2 \text{ for } n > N.$$

By using the triangle inequality, we see that

$$| D(A_n, B_n) - D(A, B) | \leqslant D(A_n, A) + D(B_n, B)$$

and the proof follows.

Lemma 3—If $x_0 \in X$ then $x_n \in Fx_{n-1}$, $n = 1, 2, \ldots$ can be so selected that

$$d(x_n, x_{n+1}) \leqslant D(Fx_{n-1}, Fx_n)$$

where $F: X \to \mathrm{Cpt}(X)$.

PROOF : Let $x_0 \in X$ and $x_1 \in Fx_0$. Then by Nadler [14], there exists $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leqslant D(Fx_0, Fx_1).$$

Since $x_2 \in Fx_1$, there exists in the same way an element $x_3 \in Fx_2$ such that

$$d(x_2, x_3) \leqslant D(Fx_1, Fx_2)$$

and in this way we can find a sequence $\{x_n\}$ in X such that

$$d(x_n, x_{n+1}) \leqslant D(Fx_{n-1}, Fx_n), \; n = 1, 2, 3, \ldots$$

where $x_n \in Fx_{n-1}$ and the proof is complete.

3. FIXED POINT THEOREM

Theorem 1—Let $F: X \to \mathrm{Cpt}(X)$ be such that

$$D(Fx, Fy) < d(x, y)\,;\; x, y \in X;\; x \neq y. \qquad \ldots(1)$$

If the sequence $\{x_n\}$ of Lemma 3 has a convergent subsequence $\{x_{n_i}\}$ say, then there exists a fixed point of F.

PROOF : If $x_n = x_{n+1}$ for some n then $x_{n+1} \in Fx_n$ implies $x_n \in Fx_n$ and hence there is a fixed point. So we suppose that $x_n \neq x_{n+1}$ for any value of n and we show that $\xi = \lim_{i\to\infty} x_{n_i} \in F\xi$.

If possible, suppose that $\xi \notin F\xi$. Let Y be the subset of $X \times X$ defined by

$$Y = X \times X - \Delta$$

where $\Delta = \{(x, y) \in X \times X : x = y\}$. We define a real-valued function of two variables on Y by

$$f(p, q) = \frac{D[Fp, Fq]}{d(p, q)}, \; (p, q) \in Y. \qquad \ldots(2)$$

Using Lemmas 1 and 2 we easily see that $f(p, q)$ is continuous on Y. Since $\xi \notin F\xi$ and $F\xi \in \mathrm{Cpt}(X)$, $d(\xi, F\xi) > 0$.

Since $x_{n_i+1} \in Fx_{n_i}$, by Nadler[14] there exists an element $\eta \in F\xi$ such that

$$d(x_{n_i+1}, \eta) \leqslant D(Fx_{n_i}, F\xi) < d(x_{n_i}, \xi) \to 0 \text{ as } i \to \infty$$

if $x_{n_i} \neq \xi$ for all large i. If, however, $x_{n_i} = \xi$ for all large i, then $x_{n_i+1} \in Fx_{n_i} = F\xi$ and because $F\xi$ is compact, there exists a subsequence of $\{x_{n_i+1}\}$ which converges to a point of $F\xi$. In this case we take η to be this point. Without loss of generality, we

may therefore assume that $x_{n_i} \neq \xi$ for all large i. By (2), $f(p, q) < 1$ for $(p, q) \in Y$ and so $f(\xi, \eta) < 1$. Since $f(p, q)$ is continuous at (ξ, η), there exists an open sphere U, say around (ξ, η) such that for $(p, q) \in U$,

$$0 \leqslant f(p, q) < R < 1. \qquad \text{...(3)}$$

There exist two open spheres $S_1 = S_1(\xi, \alpha)$ and $S_2 = S_2(\eta, \alpha)$ such that $S_1 \times S_2 \subset U$ where $0 < \alpha < 1/3\, d(\xi, \eta)$. So the open spheres S_1 and S_2 are disjoint.

Since $x_{n_i} \to \xi$ and $x_{n_i+1} \to \eta$ as $i \to \infty$, there exists a positive integer N such that

$$d(x_{n_i}, x_{n_i+1}) > \alpha \text{ for } i > N \qquad \text{...(4)}$$

and so from (3), $f(x_{n_i}, x_{n_i+1}) < R$ for $i > N$, i. e.,

$$D[Fx_{n_i}, Fx_{n_i+1}] < R\, d(x_{n_i}, x_{n_i+1}). \qquad \text{...(5)}$$

Let $1 > j > N$. A repeated use of (1) and (5) yield

$$\begin{aligned} d(x_{n_l}, x_{n_l+1}) &\leqslant D(Fx_{n_l-1}, Fx_{n_l}) < d(x_{n_l-1}, x_{n_l}) \\ &\leqslant D(Fx_{n_l-2}, Fx_{n_l-1}) < d(x_{n_l-2}, x_{n_l-1}) \\ &\leqslant \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leqslant D(Fx_{n_{l-l}}, Fx_{n_{l-l}+1}) < Rd(x_{n_{l-l}}, x_{n_{l-l}+1}) \\ &\leqslant \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leqslant RD(Fx_{n_{l-2}}, Fx_{n+1}) < R^2 d(x_{n_{l-2}}, x_{n_{l-2}+1}) \\ &\leqslant \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &< R^{l-j} d(x_{n_j}, x_{n_j+1}) \to 0 \text{ as } l \to \infty, \text{ because } R < 1. \end{aligned}$$

But this is a contradition to (4). Hence $\xi \in F\xi$ and so ξ is a fixed point of F. This completes the proof of the theorem.

4. Periodic Point Theorem

Let $F : X \to \text{Cpt}(X)$. Let $x_0 \in X$ and x_1 be an arbitrary point from Fx_0. By {Nadler[14]} there exists a point $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leqslant D(Fx_0, Fx_1).$$

Since $x_2 \in Fx_1$, similarly there exists an element $u \in Fx_2$ such that

$$d(u_1, u_2) \leqslant D(Fx_1, Fx_2). \qquad \text{...(6)}$$

Let G_2 be the collection of all such points $u \in Fx_2$ for which (6) holds. Clearly G_2 is compact and so we can find an element $u_1 \in G_2$ such that

$$d(x_1, u_1) \leqslant D(Fx_0, G_2).$$

Let $Fx_0 = G_0$. If $D(G_0, G_2) \leqslant D(Fx_0, Fx_2)$ then it may be verified that the distance between any two points of x_1, x_2, u_1 ($= x_3$, say) does not exceed the Hausdorff distance between the corresponding sets Fx_0, Fx_1, Fx_2.

Since Fx_3 is compact, there exists an element $v \in Fx_3$ such that

$$d(x_3, v) \leqslant D(Fx_2, Fx_3). \quad \text{...(7)}$$

Let G_3 be the collection of all such points $v \in Fx_3$ for which (7) holds. Since, as may be seen, G_3 is compact, there exists an element $v_1 \in G_3$ such that

$$d(x_1, v_1) \leqslant D(G_0, G_3). \quad \text{...(8)}$$

Let G_3^1 be the collection of all such points $v_1 \in G_3$ for which (8) holds. By similar reasonings, there exists an element $v_2 \in G_3^1$ such that

$$d(x_2, v_2) \leqslant D(Fx_1, G_3^1) = D(G_1, G_3^1)$$

where $Fx_1 = G_1$, say. If $D(G_0, G_3) \leqslant D(Fx_0, Fx_3)$ and $D(G_1, G_3^1) \leqslant D(Fx_1, Fx_3)$ then it may be seen that the distance between any two points of x_1, x_2, x_3, v_2 ($= x_4$, say) does not exceed the Hausdorff distance between the corresponding sets Fx_0, Fx_1, Fx_2, Fx_3.

Proceeding in this way if we suppose that $D(G_i, G_j) \leqslant D(Fx_i, Fx_j)$, $i = 0$, $j = 1, 2,$. and $D(G_i, G_j^i) \leqslant D(Fx_i, Fx_j)$, ...(A)

$i = 1, 2, 3, \ldots, j = 3, 4, \ldots$ and $i < j$, then we can construct a sequence $\{x_n\}$ such that $x_n \in Fx_{n-1}$ and

$$d(x_k, x_s) \leqslant D(Fx_{k-1}, Fx_{s-1}), \; k \neq s = 1, 2, \ldots . \quad \text{...(B)}$$

Note 1 : It may be noted that for a point-valued mapping F the relation (A) holds automatically.

We now introduce the following definitions.

Definition 1—A mapping $F : X \to B(X)$ is said to satisfy condition (C) if for each $u \in X$ there exists an open sphere $S(u, r(u))$ such that $x, y \in S(u, r(u))$ with $x \neq y$ implies $D(Fx, Fy) < d(x, y)$ and $Fx, Fy \subset S(v, r(v))$ for some $v \in X$.

Definition 2—Let $u \in X$ and let $u_k \in F\, u_{k-1}$, $u_0 = u$. Then u is said to be a periodic point of F if $u \in Fu_{k-1}$ for some $k \geqslant 1$.

For $k = 1$, u is fixed point of F.

Theorem 2—Let $F{:}X \to \mathrm{Cpt}(X)$ satisfy condition (C) and suppose that a sequence

$\{x_n\}$ has been constructed satisfying (B). If $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ then F has a periodic point.

PROOF : If for some n and k, $x_n = x_{n+k}$ then $x_n = x_{n+k} \in Fx_{n+k-1}$ and hence the theorem is obtained. So we suppose that $x_n \neq x_{n+k}$ for any n and k. Let $u = \lim_{i\to\infty} x_{n_i}$. We show that u is a periodic point of F. Because of the preceding remark, we may assume that $x_{n_i} \neq u$ for all large i. If possible, suppose that u is not a periodic point of F, i. e., $u \notin Fu_{k-1}$ for any $k \geqslant 1$, where $u_{k-1} \in Fu_{k-2}$ and $u_0 = u$.

Since F satisfies condition (C) and $x_{n_i+1} \in Fx_{n_i}$, by (Nadler[14]) there exists a point $u_1 \in Fu$ such that

$$d(x_{n_i+1}, u_1) \leqslant D(Fx_{n_i}, Fu) < d(x_{n_i}, u)$$

and $Fx_{n_i}, Fu \subset S(v, r(v))$ for some $v \in X$. Again as $x_{n_i+2} \in Fx_{n_i+1}$, there exists a point u_2 in Fu_1 such that

$$d(x_{n_i+2}, u_2) \leqslant D(Fx_{n_i+1}, Fu_1) < d(x_{n_i+1}, u_1)$$

and $Fx_{n_i+1}, Fu_1 \subset S(w, r(w))$ for some $w \in X$. By the way of induction we can find an element $u_k \in Fu_{k-1}$ such that

$$d(x_{n_i+k}, u_k) \leqslant D(Fx_{n_i+k-1}, Fu_{k-1}) < d(x_{n_i+k-1}, u_{k-1}) < \dots < d(x_{n_i}, u)$$

for all large i,

This gives that

$$\lim_{i\to\infty} d(x_{n_i+k}, u_k) = 0, \text{ for } k = 1, 2, 3, \dots \qquad \dots(9)$$

Since $\lim_{i\to\infty} x_{n_i} = u$, we select two positive integers p and k such that

$$x_p, x_{p+k} \in S(u, r(u)) \text{ and } d(x_p, x_{p+k}) < \tfrac{1}{3} r(u). \qquad \dots(10)$$

Since F satisfies condition (C), for $n = p + 1$ we have

$$d(x_n, x_{n+k}) = d(x_{p+1}, x_{p+k+1}) \leqslant D(Fx_p, Fx_{p+k})$$
$$< d(x_p, x_{p+k}) \text{ and } Fx_p, Fx_{p+k} \subset S(g, r(g))$$

for some $g \in X$. So, $x_{p+1}, x_{p+k+1} \in S(g, r(g))$ and this implies that

$$d(x_{p+2}, x_{p+k+2}) \leqslant D(Fx_{p+1}, Fx_{p+k+1}) < d(x_{p+1}, x_{p+k+1})$$

and $Fx_{p+1}, Fx_{p+k+1} \subset S(h, r(h))$ for some $h \in X$ and so

$$d(x_n, x_{n+k}) < d(x_p, x_{p+k}) \text{ for } n = p + 2.$$

By induction we can show that (10) implies

$$d(x_n, x_{n+k}) < d(x_p, x_{p+k}) \text{ for } n > p \qquad \dots(11)$$

and so

$$d(x_{n_i}, x_{n_i+k}) < d(x_p, x_{p+k}) \text{ for } n_i > p.$$

Passing on limit as $i \to \infty$, we have using (9)

$$d(u, u_k) \leqslant d(x_p, x_{p+k}),$$

i. e., $d(u, u_k) < \frac{1}{3} r(u)$ and this implies that $u_k \in S(u, \frac{1}{3} r(u))$.

Since $u \notin Fu_{k-1}$ and $u_k \in Fu_{k-1}$, we have $u \neq u_k$. Then

$$\begin{aligned} d(u_1, u_{k+1}) &\leqslant d(u_1, x_{n_i+1}) + d(x_{n_i+1}, x_{n_i+k+1}) + d(x_{n_i+k+1}, u_{k+1}) \\ &\leqslant D(Fu, Fx_{n_i}) + D(Fx_{n_i}, Fx_{n_i+k}) + D(Fx_{n_i+k}, Fu_k) \\ &\leqslant D(Fu, Fx_{n_i}) + D(Fx_{n_i}, Fu) + D(Fu, Fu_k) \\ &\quad + D(Fu_k, Fx_{n_i+k}) + D(Fx_{n_i+k}, Fu_k) \\ &< d(u, x_{n_i}) + d(x_{n_i}, u) + D(Fu_1, Fu_k) \\ &\quad + d(u_k, x_{n_i+k}) + d(x_{n_i+k}, u_k) \text{ for all large } i. \end{aligned}$$

Passing on limit as $i \to \infty$, we obtain from (9)

$$\begin{aligned} d(u_1, u_{k+1}) &\leqslant D(Fu, Fu_k) \\ &< d(u, u_k) \end{aligned}$$

and Fu, $Fu_k \subset S(f, r(f))$ for some $f \in X$. Similarly

$$d(u_2, u_{k+2}) < d(u, u_k)$$

and by induction we can show that

$$d(u_n, u_{n+k}) < d(u, u_k) \text{ for all } n. \qquad \text{...(12)}$$

We now show that $x_n \in S(u, \frac{1}{3} r(u))$ for $n > p$ implies

$$x_{n+k}, x_{n+2k} \in S(u, r(u)) \text{ for } n > p. \qquad \text{...(13)}$$

By (11) we have $d(u, x_{n+k}) \leqslant d(u, x_n) + d(x_n, x_{n+k})$

$$< \tfrac{1}{3} r(u) + d(x_p, x_{p+k}) \text{ for } n > p,$$

and

$$\begin{aligned} d(u, x_{n+2k}) &\leqslant d(u, x_{n+k}) + d(x_{n+k}, x_{n+k+k}) \\ &< d(u, x_{n+k}) + d(x_p, x_{p+k}) \text{ for } n > p. \end{aligned}$$

Hence as $d(x_p, x_{p+k}) < 1/3\ r(u)$, we obtain that

$$d(u, x_{n+k}) < 2/3\ r(u); \quad d(u, x_{n+2k}) < r(u).$$

Thus (13) is proved.

Now as $\lim_{i\to\infty} x_{n_i} = u$, we may clearly suppose than $n_i > p$ and $x_{n_i} \in S(u, 1/3\ r(u))$. Then by (13), for a fixed n_i we have $x_{n_i+k}, x_{n_i+2k} \in S(u, r(u))$. This implies, as (10) implies (11), that

$$d(x_n, x_{n+k}) < d(x_{n_i+k}, x_{n_i+2k}) \text{ for all } n > n_i + k.$$

Hence as $d(u, u_k)$ is a limit point of $\{d(x_n, x_{n+k})\}_{n=0}^{\infty}$

$$d(u, u_k) \leqslant d(x_{n_i+k}, x_{n_i+2k}).$$

Passing on limit as $i \to \infty$, we have

$$d(u, u_k) \leqslant d(u_k, u_{2k})$$

which contradicts (12) for $n = k$. This shows that $u \in Fu_{k-1}$ for some k where $u_0 = u$. Hence u is a periodic point of F and this completes the proof of the theorem.

Corollary 1[5]—Let $F: X \to \text{Cpt}(X)$ be such that $D(Fx, Fy) < d(x, y), x, y \in X$, $x \neq y$ and whenever $d(x, y) < \epsilon$, $\epsilon > 0$ and suppose that a sequence has been constructed satisfying (B). If $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ then F has a periodic point.

The Corollary is a genaralisation of Edelstein's theorem on point valued mapping.

5. Fixed Point from Periodic Point

In this section we shall examine when a periodic point of a set valued mapping F satisfying the condition (C) is a fixed point of F. For this we require the notion of convexity in the sence of Menger[1]. A subset M of X is said to be convex if for every pair of points $x, y \in M$ there exists a point $z \in M$ such that $d(x, y) = d(x, z) + d(z, y)$. Menger's theorem[1] states that a convex and complete metric space contains together with x, y also a metric segment whose extremities are x and y; in other words a subset isometric to an interval of length $d(x, y)$.

Theorem 3—Let X be a complete convex metric space and let $F: X \to \text{Cpt}(X)$ satisfy condition (C). Let $\{x_n\}$ be a sequence satisfying (B). If $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ then F has a fixed point.

Proof : It follows from Theorem 2 that F has a periodic point, say u in X. We suppose that u is not a fixed point of F. i. e., $u \notin Fu$. Since Fu is compact, there exists an element $u_1 \in Fu$ such that

$$d(u, u_1) = \inf_{x \in Fu} d(u, x) > 0. \qquad \ldots(14)$$

Since X is complete and convex, by Menger's theorem[1] there exists a metric interval $[u_1, u]$ with the end points u_1, u and isometric to an interval of length $d(u_1, u)$ and so $[u_1, u]$ is compact. Therefore there exists an $\epsilon > 0$ such that for any $x, y \in [u_1, u]$ with $d(x, y) < \epsilon$, there exists a sphere $S(z, r(z))$, say where $x, y \in S(z, r(z))$ for some $z \in [u_1, u]$. We now take the metric interval

$$u_1 = x_0, x_1, x_2, \ldots, x_{m-1}, x_m = u$$

with $d(x_{j-1}, x_j) < \epsilon$ for $j = 1, 2, \ldots, m$

and

$$d(u_1, u) = \sum_{j=1}^{m} d(x_{j-1}, x_j). \qquad \ldots(15)$$

Since F satisfies condition (C),

$$D(Fx_{j-1}, Fx_j) < d(x_{j-1}, x_j), \; j = 1, 2, \ldots, m$$

and $Fx_{j-1}, Fx_j \subset S(w, r(w))$ for some $w \in X$.

Starting from a point $x_0^1 \in Fx_0$, we obtain by using[14] successively the points $x_r^1 \in Fx_r$, $r = 1, 2, \ldots, m$ such that

$$d(x_{j-1}^1, x_j^1) \leqslant D(Fx_{j-1}, Fx_j), \; j = 1, 2, \ldots, m \qquad \ldots(16)$$

and x_j^1 in Fx_j depends on the choice of x_{j-1}^1 in Fx_{j-1}, $j = 1, 2, \ldots, m$. Similarly, starting from a point x_0^2 in Fx_0^1, we obtain the inequalities

$$d\left(x_{j-1}^2, x_j^2\right) \leqslant D\left(Fx_{j-1}^1, Fx_j^1\right) \qquad \ldots(17)$$

and x_j^2 in Fx_j^1 depends on the choice of x_{j-1}^2 in Fx_{j-1}^1, $j = 1, 2, \ldots, m$.

In the same way for a positive integer $k > 2$ we can obtain a set of finite points

$$x_0^{k-1}, x_1^{k-1}, \ldots, x_m^{k-1} \qquad \ldots(18)$$

respectively from

$$Fx_0^{k-2}, Fx_1^{k-2}, \ldots, Fx_m^{k-2}$$

such that

$$d\left(x_{j-1}^{k-1}, x_j^{k-1}\right) \leqslant D\left(Fx_{j-1}^{k-2}, Fx_j^{k-2}\right) \qquad \ldots(19)$$

and x_j^{k-1} in Fx_j^{k-2} depends on the choice of x_{j-1}^{k-1} in Fx_{j-1}^{k-2}, $j = 1, 2, \ldots, m$. Since u is a periodic point, it follows that $u \in Fx_0^{k-2}$ for some $k > 2$. In (18) we may therefore assume that $u = x_0^{k-1}$. Also, since u is a periodic point, it follows that $u \in Fx_m^{k-1}$. It is clear that $u_1 \in Fx_0^{k-1} = Fu$ (since $x_0^{k-1} = u$). Since $u \in Fx_m^{k-1}$, there exists a point x_{m-1}^k in Fx_{m-1}^{k-1} such that

$$d\left(u, x_{m-1}^k\right) \leqslant D\left(Fx_m^{k-1}, Fx_{m-1}^{k-1}\right)$$

i. e.

$$d\left(x_m^k, x_{m-1}^k\right) \leqslant D\left(Fx_{m-1}^{k-1}, Fx_{m-1}^{k-1}\right), \text{ where } x_m^k = u,$$

Again since x_{m-1}^k is in Fx_{m-1}^{k-1}, there exists a point x_{m-2}^k in Fx_{m-2}^{k-1} such that

$$d\left(x_{m-1}^k, x_{m-2}^k\right) \leqslant D\left(Fx_{m-1}^{k-1}, Fx_{m-2}^{k-1}\right)$$

and in this way there exists a point x_0^k in Fx_0^{k-1} ($= Fu$) such that

$$d\left(x_1^k, x_0^k\right) \leqslant D\left(Fx_1^{k-1}, Fx_0^{k-1}\right),$$

i.e. is short,

$$d\left(x_j^k, x_{j-1}^k\right) \leqslant D\left(Fx_j^{k-1}, Fx_{j-1}^{k-1}\right) \qquad \ldots(20)$$

and the choice of the point x_{j-1}^k in Fx_{j-1}^{k-1} depends on the choice of x_j^k in Fx_j^{k-1}, $j = 1, 2, \ldots, m$; $k > 2$. Since $x_0^k \in Fu$,

$$d(u, u_1) \leqslant d(u, x_0^k) \text{ by (14).}$$

So,

$$d(u_1, u) = d(u, u_1) \leqslant d(u, x_0^k) = d\left(x_m^k, x_0^k\right)$$
$$\leqslant d\left(x_m^k, x_{m-1}^k\right) + d\left(x_{m-1}^k, x_{m-2}^k\right) \ldots +$$

(equation continued on p. 727)

$$+ \dots + d\left(x_1^k, x_0^k\right) \leqslant D\left(Fx_m^{k-1}, Fx_{m-1}^{k-1}\right) + D\left(Fx_{m-1}^{k-1},\right.$$

$$\left. Fx_{m-2}^{k-1}\right) + \dots + D\left(Fx_1^{k-1}, Fx_0^{k-1}\right) \text{ by (20)}$$

$$< d\left(x_m^{k-1}, x_{m-1}^{k-1}\right) + d\left(x_{m-1}^{k-1}, x_{m-2}^{k-1}\right) + \dots$$

$$+ d\left(x_1^{k-1}, x_0^{k-1}\right), \text{ since } F \text{ satisfies (C)}$$

$$= d\left(x_0^{k-1}, x_1^{k-1}\right) + \dots + d\left(x_{m-2}^{k-1}, x_{m-1}^{k-1}\right)$$

$$+ d\left(x_{m-2}^{k-1}, x_m^{k-1}\right) \leqslant D\left(Fx_0^{k-2}, Fx_1^{k-2}\right) + \dots$$

$$+ D\left(Fx_{m-2}^{k-2}, Fx_{m-1}^{k-2}\right) + D\left(Fx_{m-1}^{k-2}, Fx_m^{k-2}\right) \text{ by (19)}$$

$$< \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\leqslant D(Fx_0, Fx_1) + D(Fx_1, Fx_2) + \dots + D(Fx_{m-2}, Fx_{m-1})$$

$$+ D(Fx_{m-1}, Fx_m), \text{ by (16)}$$

$$< d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{m-2}, x_{m-1})$$

$$+ d(x_{m-1}, x_m) = d(u_1, u), \text{ by (15)}$$

which is not possible, Thus we arrive at a contradiction and hence $u \in Fu$, i. e., u is a fixed point of F. This completes the proof of the theorem.

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ON A GENERAL CLASS OF ABSTRACT FUNCTIONAL INTEGRODIFFERETIAL EQUATIONS

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The aim of the present paper is to study the existence, uniqueness, stability, boundedness and asymptotic behavior of solutions of functional integrodifferential equations of the more general type in a Banach space. The method of successive approximation, comparison theorems and integral inequality of the more general type are used to establish our results.

1. INTRODUCTION

Let X be a Banach space with norm $\|.\|$ and $B(X)$ denotes the Banach space of all bounded linear maps on X, with the norm of uniform operator topology. Let R^+ denote the non-negative real line, N denote the set of natural numbers and $J_0 = [0.T]$, $J = [-r, T]$, $T > 0$, $r > 0$ are constants. The sets of all continuous mappings from a topological space S into X and from a topological space S into R^+ are denoted by $C(S; X)$ and $C(S; R^+)$ respectively. If $S = [a, b]$, $a < b$ then these sets are endowed with the uniform topology. We denote the uniform norm on $C = C([-r, 0], X)$ by $\|.\|_0$. If $x \in C(J; X)$ and $t \in J_0$ then x_t denotes the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. Consider the functional integrodifferential equation of the form

$$\left.\begin{aligned} & x'(t) + A(t)\,x(t) = f\Big(t, x_t, \int_0^t k(t, s, x_s)\,ds\Big),\ t \in J_0 \\ & x_0 = \phi(t),\ -r \leqslant t \leqslant 0 \end{aligned}\right\} \qquad \ldots(1.1)$$

in a Banach space X where $\phi \in C$ is given, $x : J \to X$, $-A(t)$, for each $t \in J_0$, is the infinitesimal generator of analytic semigroup, $k : J_0 \times J_0 \times C \to X$ and $f : J_0 \times C \times X \to X$ are continuous.

The equations of this type commonly come across in almost all phases of physics and other areas of applied mathematics, see Belleni-Morante[1], Burch and Goldetein[2], Friedman[3], Nohel[12] and references listed therein. Many recent papers and monographs have dealt with the existence, uniqueness and other properties of solutions of special forms of eqn. (1.1)[4-9,14-16,18,19]. Kartsatos and Parrott[8] have proved the existence of a unique strong solution of eqn. (1.1) when $k = 0$ and for each $t \in J_0$, $A(t)$ is m-accretive operator, by using the method of lines. Heikkila[6] has established the existence and

uniqueness of mild and strict solutions of the equation (1.1) when $k = 0$ by using the method of successive approximation and comparison principle. In a series of papers Pachpatte [14-16] has established several interesting results regarding the boundedness, stability and asymptotic behaviour of the solutions of equations of the form (1.1) without functional arguments when $A(t)$, for each $t \in R^+$, is linear closed operator by assuming the existence of solutions and by using comparison principle and the integral inequality of the more general type given by Pachpatte[13]. In view of the general form of (1.1), the study of the problems of existence, uniqueness and other properties of the solutions of (1.1) are quite interesting and deserves special attention.

The main objective of the present paper is to study the existence, uniqueness, boundedness, stability and asymptotic behaviour of solutions of equation (1.1). The main tools employed in our analysis are based on the method of successive approximation, comparison theorems and the integral inequality recently established by Pachpatte[13]. In fact, our results in the present paper are motivated by the recent papers of Heikkila[5,6] and Pachpatte[14].

⇔ The paper is organised as follows.

In section 2, we set up preliminaries and give the statement of the main results, the proofs of which are contained in sections 3, 4 and 5. In section 6, we give some applications to illustrate some of our results established in sections 3, 4 and 5.

2. Preliminaries and Statement of Results

Before proceeding to the statement of our main results, we shall set-forth some preliminaries from Heikkila[6], Pachpatte[14] and hypotheses that will be used in our subsequent discussion.

First, we shall study the following integral equation

$$x(t) = u(t) + \int_0^{t^+} U(t^+, s)\,[f(s, x_s, \int_0^s k(s, \tau, x_\tau)\,d\tau)]\,ds, \quad t \in J \qquad \text{..(2.1)}$$

where $u(t) \in C(J, X)$, $t^+ = \max\{t, 0\}$ and $U : J_0 \times J_0 \to B(X)$ is strongly continuous. i. e. $(t, s) \to U(t, s)\,x$ is continuous for each $x \in X$. By virtue of The Uniform Boundedness Theorem, the constant M given by

$$M = \sup\{\|U(t, s)\| : 0 \leqslant s \leqslant t \leqslant T\} \qquad \text{..(2.2)}$$

is finite. For $x_1, x_2 \in C(J_0, R^+)$, define $x_1 \leqslant x_2$ if and only if $x_1(t) \leqslant x_2(t)$, $t \in J_0$. Given $k \in C(J_0 \times J_0 \times C, X)$, $f \in C(J_0 \times C \times X, X)$ we define a mapping F on $C(J, X)$ by

$$Fx(t) = \int_0^{t^+} U(t^+, s)\,f(s, x_s, \int_0^s k(s, \tau, x_\tau)\,d\tau)\,ds, \quad t \in J. \qquad \text{...(2.3)}$$

Using this definition of F, the integral equation (2.1) can be expressed symbolically as

$$x = u + Fx. \qquad \text{...(2.4)}$$

Let $\{A(t) : t \in J_0\}$ be a family of closed linear operators from a dense subset of X into X. Suppose that

(C_1) there exist $\alpha \in (\frac{\pi}{2}, x)$ and $K > 0$ such that $[\lambda I + A(t)]^{-1}$ exists, belongs to $B(X)$ and $\|[\lambda I + A(t)]^{-1}\| \leqslant \frac{K}{|\lambda| + 1}$ whenever $t \in J_0$ and λ is a complex number for which $|\arg \lambda| \leqslant \alpha$ (Here I denotes the identity operator of X).

(C_2) $t \to A(t)(A)s))^{-1}$ is, for each $s \in J_0$ a uniformly Holder continuous mapping from J_0 into $B(X)$, the Holder constant and exponent being independent of $s \in J_0$.

It is easy to observe that the operators—$A(t)$ form a parabolic family i. e. $-A(t)$, for each $t \in J$, is the infinitesimal generator of an analytic semigroup (see Laddas and Lakshmikantham[9], p. 59, Remark 3. 1. 1). From above conditions (C_1) and (C_2), it follows that the evolution equation

$$u'(t) + A(t)u(t) = 0 \qquad \text{...(2.5)}$$

has a unique fundamental solution $U : J_0 \times J_0 \to B(X)$ and that for each $x_0 \in X$ the mapping $t \to U(t, 0)x_0$ is the unique solution of (2.5) with value x_0 at $t = 0$ (see Ladas and Lakshmikantham[9], p. 60, Theorems 3.21). Now, the mapping $u \in C(J, X)$ defined by

$$u(t) = U(t^+, 0)\,\phi(t^-) \text{ for } t \in J \qquad \text{...(2.6)}$$

where $t = \min\{t, 0\}$, is the solution of (2.5) with the given $\phi \in C$ as the initial mapping.

We need the following definitions in out subsequent discussion [see Hsikkila[6] p. 7 and 9), Lakshmikantham and Leela[10], p. 21 and 49)., Pachpatte (p. 339).

Definition 1—A solution of the integral equation

$$x(t) = U(t^+, 0)\,\phi(t^-) + \int_0^{t^+} U(t^+, s)\,[f(s, x_s, \int_0^s k(s, \tau, x_\tau)\,d\tau)]\,ds,\ t \in J \qquad \text{...(2.7)}$$

is called a mild solution of (1.1).

Definition 2—A continuous mapping $x : J \to X$ is called a strict solution of (1.1) if x equals to ϕ on $[-r, 0]$ and is strongly continuously differentiable and satisfies (1.1) on $(0, T]$.

Definition 3—The solution $x(t)$ of (1.1) is said to be exponentially asymptotically stable, if there exist positive constants Q and β such that the inequality

$$\|x_t\|_0 \leqslant Q\,\|\phi\|_0\, e^{-\beta t},\ t \geqslant 0$$

holds for $\|\phi\|_0$ sufficiently small.

Definition 4—The solution $x(t)$ of (1.1) is said to be uniformly slowly growing if, and only if, for every $\beta > 0$ there exists a constant Q, possibly depending on β, such that the inequality

$$\|x_t\|_0 \leqslant Q\,\|\phi\|_0\, e^{\beta t},\ t \geqslant 0$$

holds for $\|\phi\|_0 < \infty$.

For our convenience we list the following hypotheses.

(H_1) For (t, s, ψ), $(t, s, \bar{\psi}) \in J_0 \times J_0 \times C$,

$$\|k(t, s, \psi) - k(t, s, \bar{\psi})\| \leqslant q(t, s, \|\psi - \bar{\psi}\|_0) \quad \text{...(2.8)}$$

where $q \in C(J_0 \times J_0 \times R^+, R^+)$ and $q(t, s, m)$ is nondecreasing in $m \in R^+$, for each $(t, s) \in J_0 \times J_0$.

(H_2) For (t, ψ, y), $(t, \bar{\psi}, \bar{y}) \in J_0 \times C \times X$

$$\|f(t, \psi, y) - f(t, \bar{\psi}, \bar{y})\| \leqslant p(t, \|\psi - \bar{\psi}\|_0, \|y - \bar{y}\|) \quad \text{...(2.9)}$$

where $p \in C(J_0 \times R^+ \times R^+, R^+)$ and $p(t, m_1, m_2)$ is nondecreasing in $m_1, m_2 \in R^+$, for each fixed $t \in J_0$.

(H_3) For each $h \in C(J_0, R^+)$, the integral equation

$$w(t) = h(t) + M \int_0^t p(s, w(s), \int_0^s q(s, \tau, w(\tau))\, d\tau)\, ds \quad \text{...(2.10)}$$

has a solution on J_0.

(H_4) Assume that $w(t) \equiv 0$ is the only solution of (2.10) with $h(t) = 0$.

(H_5) For $v \in C(J, X)$, $t, s \in J_0$, $\psi \in C$, $y \in X$,

$$\|k(t, s, \psi)\| \leqslant q^*(t, s, \|\psi - v_s\|_0) \quad \text{...(2.11)}$$

$$\|f(t, \psi, y)\| \leqslant p^*(t, \|\psi - v_t\|_0, \|y\|) \quad \text{...(2.12)}$$

where $q^* \in C(J_0 \times J_0 \times R^+, R^+)$, $q^*(t, s, m)$ is nondecreasing in $m \in R^+$, for each $(t,s) \in J_0 \times J_0$; $p^* \in C(J_0 \times R^+ \times R^+, R^+)$, $p^*(t, m_1, m_2)$ is nondecreasing in $m_1, m_2 \in R^+$ for each fixed $t \in J_0$ and for each $h \in C(J_0, R^+)$, the integral equation

$$w(t) = h(t) + M \int_0^t p^*(s, w(s), \int_0^s q^*(s, \tau, w(\tau))\, d\tau)\, ds$$

has a solution on J_0.

(H_6) For $t, s \in R^+$, $\|U(t, s)\| \leqslant M$,

where M is nonnegative constant.

(H_7) For $t, s \in R^+$, $\psi \in C$, $y \in X$

$$\|k(t, s, \psi)\| \leqslant L_1(s)\, \|\psi\|_0 \quad \text{...(2.13)}$$

$$\|f(t, \psi, y)\| \leqslant L_2(t)\, [\|\psi\|_0 + \|y\|] \quad \text{...(2.14)}$$

where $L_1, L_2 \in C(R^+, R^+)$ and

$$\int_0^\infty L_1(s)\,ds < \infty, \quad \int_0^\infty L_2(s)\,ds < \infty. \qquad \text{...(2.15)}$$

(H_8) For $t, s \in R^+$, $\beta > 0$, $M \geqslant 0$,

$$\|U(t, s)\| \leqslant M e^{-\beta(t-2s)} \qquad \text{...(2.16)}$$

(H_9) For $t, s \in R^+$, $\beta > 0$, $\psi \in C$, $y \in X$,

$$\|k(t, s, \psi)\| \leqslant e^{-\beta(t-s)} L_1(s) \|\psi\|_0 \qquad \text{...(2.17)}$$

$$\|f(t, \psi, y)\| \leqslant e^{-\beta t} L_2(t) [\|\psi\|_0 + \|y\|] \qquad \text{...(2.18)}$$

where $L_1, L_2 \in C(R^+, R^+)$ and (2.15) holds

(H_{10}) For $t, s \in R^+$, $\beta > 0$, $M \geqslant 0$,

$$\|U(t, s)\| \leqslant M e^{\beta(t-2s)} \qquad \text{...(2.19)}$$

(H_{11}) For $t, s \in R^+$, $\beta > 0$, $\psi \in C$, $y \in X$,

$$\|k(t, s, \psi)\| \leqslant e^{\beta(t-s)} L_1(s) \|\psi\|_0 \qquad \text{...(2.20)}$$

$$\|f(t, \psi, y)\| \leqslant e^{\beta t} L_2(t) [\|\psi\|_0 + \|y\|] \qquad \text{...(2.21)}$$

where $L_1, L_2 \in C(R^+, R^+)$ and (2.15) holds.

We require the following integral inequality established by Pachpatte[13] to prove some of our results.

Lemma 1 (see Pachpatte[13], p. 758)—Let $a(t)$, $b(t)$ and $c(t)$ be real-valued non-negative continuous functions defined on R^+, for which the inequality

$$c(t) \leqslant c_0 + \int_0^t a(s) c(s)\,ds + \int_0^t a(s) \left[\int_0^s b(\tau) c(\tau)\,d\tau\right] ds,$$

holds for all $t \in R^+$, where c_0 is a non-negative constant. Then

$$c(t) \leqslant c_0 \left[1 + \int_0^t a(s) \exp\left[\int_0^s (a(\tau) + b(\tau))\,d\tau\right] ds\right],$$

for all $t \in R^+$.

With these preparations we are now in a position to state our main results to be proved in this paper.

Theorem 1—Assume that the hypotheses (H_1) — (H_4) hold. Then for each $u \in C(J, X)$, the successive approximations defined by

$$x^{n+1} = u + Fx^n, \; n \in N \qquad \text{...(2.22)}$$

with any $x^1 \in C(J, X)$ as the first approximation, converge on J uniformly to a unique solution of (2.1).

Theorem 2—In addition to the hypotheses of Theorem 1, if for $v \in C(J, X)$, $h \in C(J_0, R^+)$,

$$\|v_t - u_t - Fv_t\|_0 \leqslant h(t),\ t \in J_0 \qquad \text{...(2.23)}$$

is satisfied then

$$\|v_t - x_t\|_0 \leqslant w(t),\ t \in J_0 \qquad \text{...(2.24)}$$

where $w(t)$ is the minimal solution of (2.10).

As an immediate consequence of Theorem 2, we have the following corollary.

Corollary 1—Suppose that the hypotheses of Theorem 1 hold. If x and $\bar{x}$ are two solutions of (2.1) with given u and $\bar{u} \in C(J, X)$ respectively then

$$\|x_t - \bar{x}_t\|_0 \leqslant w(t),\ t \in J_0 \qquad \text{...(2.25)}$$

where w is the minimal solution of (2.10) with $h(t) = \|u_t - \bar{u}_t\|_0$.

Theorem 3—Assume that the hypotheses $(H_1) - (H_5)$ hold. Then

$$\|x_t - v_t\|_0 \leqslant z(t),\ t \in J_0 \qquad \text{...(2.26)}$$

where $z(t)$ is the minimal solution of

$$z(t) = \|u_0 - v_t\|_0 + M \int_0^t p^*\left(s, z(s), \int_0^s q^*(s, \tau, z(\tau))\, d\tau\right) ds. \qquad \text{...(2.27)}$$

Theorem 4—Suppose that f depends on a parameter belonging to a nonempty set E, and the hypotheses of Theorem 1 for given $\mu, \bar{\mu} \in E$ hold. If x and $\bar{x}$ denote the corresponding solutions of (2.1) with the same $u \in C(J, x)$ then

$$\|x_t - \bar{x}_t\|_0 \leqslant w(t),\ t \in J_0 \qquad \text{...(2.28)}$$

where w is the minimal solution of (2.10) with

$$h(t) = M \int_0^t \Big\| f\Big(s, \bar{x}_s, \int_0^s k(s, \tau, \bar{x}_\tau)\, d\tau, \bar{\mu}\Big)$$

$$- f\Big(s, \bar{x}_s, \int_0^s k(s, \tau, \bar{x}_\tau)\, d\tau, \mu\Big)\Big\|\, ds. \qquad \text{...(2.29)}$$

Theorem 5—Suppose that the hypotheses (H_1) and (H_2) hold, and the equation

$$\left.\begin{aligned} w'(t) &= Mp\left(t, w(t), \int_0^t q(t, s, w(s))\, ds\right), \\ w(0) &= w_0 \end{aligned}\right\} \qquad \text{...(2.30)}$$

has a solution on J_0 for each $w_0 \geqslant 0$. Further suppose that $w(t) = 0$ is the only solution of (2.30) with $w_0 = 0$. Then for each $\phi \in C$, the equation (1.1) has a unique mild solution x on J. Moreover, x depends continuously on ϕ.

Theorem 6—Let E be a metric space. Suppose that $f \in C(J_0 \times C \times X \times E, X)$ and the hypotheses of Theorem 5 hold for each fixed $\mu \in E$. Then the mild solution of (1.1) depends continuously on the parameter μ.

Theorem 7—Let $\{A(t) : t \in J_0\}$ be a family of operators in $B(X)$ satisfying the conditions (C_1) and (C_2) and uniformly bounded over $t \in J_0$. Further suppose that the hypotheses of Theorem 5 hold. Then for each $\phi \in C$ eqn. (1.1) has a unique strict solution on J.

Remark 1 : We note that Kartsatos and Parrott[7] have established the unique strong solution of eqn. (1.1) when $k = 0$ with assumptions that for each $t \in J_0$, $A(t)$ is m-accretive operator and f Lipschitzian like function by using different approach for evolution equations. Here our conditions on functions involved in (1.1) and the approach to the problem are different.

In the following theorems, we assume that solutions of eqn. (1.1) exist on $[-r, \infty)$.

Theorem 8—Let the hypotheses (H_6) and (H_7) be satisfied. Then any solution $x(t)$ of (1.1) is bounded on $[-r, \infty)$.

Theorem 9—Let the hypotheses (H_8) and (H_9) be satisfied. Then any soultion $x(t)$ of (1.1) is exponentially asymptotically stable.

Theorem 10—Let the hypotheses (H_{10}) and (H_{11}) be satisfied. Then any solution $x(t)$ of (1.1) is uniformly slowly growing.

Remark 2 : It is to be noted that Pachpatte[14] has studied the problems of boundedness, stability, asymptotic behaviour and other properties of the solutions of (1.1) without functional arguments when for each $t \in R^+$, $A(t)$ is a linear closed operator. Here our conditions on functions involved in (1.1) and the approach to the problem are different.

3. Proofs of Theorems 1-4 and Corollary 1

We first prepare a lemma which will be used for the proofs of Theorems 1-4.

Lemma 2—Let $q(t, s, m) \in C(J_0 \times J_0 \times R^+, R^+)$ and $p(t, m_1, m_2) \in C(J_0 \times R^+ \times R^+, R^+)$ and monotone nondecreasing in m for each $(t, s) \in J_0 \times J_0$ and in m_1, m_2 for each $t \in J_0$ respectively. Suppose that the hypothesis (H_3) holds. Let $h \in C(J_0, R^+)$ be given. Then eqn. (2.10) has a minimal solution w. If $\{u^n\}$ is a sequence in $C(J, X)$ such that $u^n \to u$ uniformly on J and if for each $t \in J_0$

$$\left.\begin{aligned} &\|u_t^1\|_0 \leqslant h(t) \\ &\|u_t^{n+1}\|_0 \leqslant h(t) + M \int_0^t p\left(s, \|u_s^n\|_0, \int_0^s q(s, \tau \|u_\tau^n\|_0)\, d\tau\right) ds,\ n \in N \end{aligned}\right\} \quad \ldots(3.1)$$

then

$$\|u_t\|_0 \leqslant w(t), \ t \in J_0. \quad \ldots(3.2)$$

PROOF : In view of the hypothesis (H_3), we see that the integral equation

$$z(t) = z_0 + M \int_0^t p(s, z(s), \int_0^s q(s, \tau, z(\tau))\, d\tau)\, ds \quad \ldots(3.3)$$

has a solution on J_0 for each $z_0 \in R^+$. Define a sequence $\{w_n\}$ inductively as

$$\left.\begin{aligned} w_1(t) &= h(t) \\ w_{n+1}(t) &= w_1(t) + M \int_0^t p(s, w_n(s), \int_0^s q(s, \tau, w_n(\tau))\, d\tau)\, ds \end{aligned}\right\} \quad \ldots(3.4)$$

$t \in J_0$ and $n \in N$. It is easy to observe that the sequence $\{w_n\}$ is nondecreasing on J_0 i. e.

$$w_n(t) \leqslant w_{n+1}(t), \text{ for all } t \in J_0, n \in N. \quad \ldots(3.5)$$

From (3.4) and (3.5) we obtain

$$w_{n+1}(t) \leqslant w_1(t) + M \int_0^t p(s, w_{n+1}(s), \int_0^s q(s, \tau, w_{n+1}(\tau))\, d\tau)\, ds \quad \ldots(3.6)$$

for $t \in J_0$ and $n \in N$. Choose $z_0 > \max\limits_{0 \leqslant t \leqslant T} w_1(t)$. Using (3.3), (3.6) and modified version of comparison Theorem (see, Lemma 1, Pachpatte[17]) we get

$$w_{n+1}(t) \leqslant z(t), \ t \in J_0 \text{ and } n \in N.$$

Thus, the sequence $\{w_n\}$ is nondecreasing and bounded above by a function $z(t)$, Hence the sequence $\{w_n\}$ converges uniformly on J_0 and $w(t) \underline{\underline{\text{def}}} \lim\limits_{n \to \infty} w_n(t)$ is a solution of (2.10).

We now prove that the limit function $w(t)$ is the minimal solution of (2.10). Let $\bar{w}$ be any solution of (2.10). By induction, we can show that

$$w_n(t) \leqslant \bar{w}(t), \ t \in J_0 \text{ and } n \in N. \quad \ldots(3.7)$$

By taking limit in (.7) as $n \to \infty$ we get $w(t) \leqslant \bar{w}(t)$, $t \in J_0$ which implies the minimality of the solution $w(t)$.

Using (3.1) and the fact that $w(t)$ is the minimal solution of (2.10), we obtain by induction that

$$\|u_t^n\|_0 \leqslant w(t), \ t \in J_0 \text{ and } n \in N. \quad \ldots(3.8)$$

Since $u^n \to u$, $u_t^n \to u_t$ we get (3.2) from (3.8) and the proof of the Lemma 2 is complete.

In order to prove Theorem 1, let $x^1 \in C(J, X)$ be given. Using (2.3), (2.8) and (2.9) we obtain

$$\|Fx(t) - F\bar{x}(t)\|$$

$$\leqslant M \int_0^{t^+} p(s, \|x_s - \bar{x}_s\|, \int_0^s q(s, \tau, \|x_\tau - \bar{x}_\tau\|_0 d\tau)\, ds. \quad \ldots(3.9)$$

$t \in J$, whenever $x, \bar{x} \in C(J, X)$, and $t^+ = \max\{t, 0\}$.

Case 1—Suppose $t \geqslant r$. Then for any $\theta \in [-r, 0]$ we have $t + \theta \geqslant 0$. For such θ' s, from (3.9), we have

$$\|Fx(t+\theta) - F\bar{x}(t+\theta)\|$$

$$\leqslant M \int_0^{t+\theta} p(s, \|x_s - \bar{x}_s\|_0, \int_0^s q(s, \tau, \|x_\tau - \bar{x}_\tau\|_0)\, d\tau)\, ds$$

$$\leqslant M \int_0^{t} p(s, \|x_s - \bar{x}_s\|_0, \int_0^s q(s, \tau, \|x_\tau - \bar{x}_\tau\|_0)\, d\tau)\, ds$$

which yields

$$\|Fx_t - F\bar{x}_t\|_0$$

$$\leqslant M \int_0^{t} p(s, \|x_s - \bar{x}_s\|_0, \int_0^s q(s, \tau, \|x_\tau - \bar{x}_\tau\|)_0)\, d\tau)\, ds \quad \ldots(3.10)$$

where Fx_t means $(Fx)_t$.

Case 2—Suppose $0 \leqslant t < r$. Then for all $\theta \in [-r, -t)$ we have $t + \theta < 0$, For such θ's we observe that

$$\|Fx(t+\theta) - F\bar{x}(t+\theta)\| \leqslant 0. \quad \ldots(3.11)$$

For $\theta \in [-t, 0]$, $t + \theta \geqslant 0$. Then, from (3.9), we get as in Case 1.

$$\|Fx(t+\theta) - F\bar{x}(t+\theta)\|$$

$$\leqslant M \int_0^{t} p(s, \|x_s - \bar{x}_s\|, \int_0^s q(s, \tau, \|x_\tau - \bar{x}_\tau\|_0)\, d\tau)\, ds. \quad \ldots(3.12)$$

Thus, for every $\theta \in [-r, 0]$, $(0 \leqslant t < r)$ from (3.11) and (3.12) we have

$$\|Fx(t+\theta) - F\bar{x}(t+\theta)\|$$

$$\leqslant M \int_0^{t} p(s, \|x_s - \bar{x}_s\|_0, \int_0^s q(s, \tau, \|x_\tau - \bar{x}_\tau\|_0)\, d\tau)\, ds$$

which implies

$$\|Fx_t - F\bar{x}_t\|_0 \leqslant M \int_0^t p\,(s, \|x_s - \bar{x}_s\|_0, \int_0^s q\,(s, \tau, \|x_\tau - \bar{x}_\tau\|_0)\, d\tau)\, ds. \qquad \ldots(3.13)$$

For every $t \in J_0$, from (3.10) and (3.13) we have

$$\|Fx_t - F\bar{x}_t\|_0 \leqslant M \int_0^t p\,(s, \|x_s - \bar{x}_s\|_0, \int_0^s q\,(s, \tau, \|x_\tau - \bar{x}_\tau\|_0)\, d\tau)\, ds. \qquad \ldots(3.14)$$

For each $n \in N$ and $t \in J_0$, we have

$$\begin{aligned} \|x_t^{n+1} - x_t^1\|_0 &\leqslant \|u_t - x_t^1\|_0 + \|Fx_t^1\|_0 + \|Fx_{n_t} - F x_t^1\|_0 \\ &\leqslant h\,(t) + M \int_0^t p\,(s, \|x_s^n - x_s^1\|_0, \int_0^s q\,(s, \tau, \|x_v^n - x_\tau^1\|_0)\, d\tau)\, ds \end{aligned} \qquad \ldots(3.15)$$

where

$$h\,(t) = \|u_t - x_t^1\|_0 + M \int_0^t \|\, f\,(s, x_s^1, \int_0^s k\,(s, \tau, x_\tau^1)\, d\tau)\|\, ds. \qquad \ldots(3.16)$$

Using monotone nondecreasing character of functions q and p, we obtain, from (3.15), by induction that

$$\|x_t^{n+1} - x_t^1\|_0 \leqslant w\,(t),\ t \in J_0,\ n \in N \qquad \ldots(3.17)$$

where w is any solution of (2.10) with h is given by (3.16).

To prove the convergence of the sequence $\{x^n\}$ on J, we define a sequence $\{z_n\}$ inductively by

$z_1\,(t) = w\,(t) =$ the minimal solution of (2.10) with h given by (3.16),

and

$$z_{n+1}\,(t) = M \int_0^t p\,(s, z_n\,(s), \int_0^s q\,(s, \tau, z_n\,(\tau))\, d\tau)\, ds. \qquad \ldots(3.18)$$

In view of (3.17), we claim that the inequality

$$\|x_t^{n+m} - x_t^m\|_0 \leqslant z_m(t) \qquad \ldots(3.19)$$

holds for $m = 1, n \in N$ and $t \in J_0$. From (3.14) it follows that

$$\|Fx_t^{n+m} - Fx_t^m\|_0$$

$$\leqslant M \int_0^t p\left(s, \|x_s^{n+m} - x_s^m\|_0, \int_0^s q(s, \tau, \|x_\tau^{n+m} - x_\tau^m\|)d\tau\right) ds \qquad \ldots(3.20)$$

whenever $m, n \in N$ and $t \in J_0$. By virtue of (3.20), we can easily verify by induction that (3.19) holds for all $m, n \in N$ and $t \in J_0$. In particular, we have

$$\|x^{n+m}(t+\theta) - x^m(t+\theta)\|$$

$$\leqslant \|x_t^{n+m} - x_t^m\|_0 \leqslant z_m(t) \leqslant z_m(T). \qquad \ldots(3.21)$$

for all $n, m \in N$, $\theta \in [-r, 0]$ and $t \in J_0$.

The sequence $\{z_n\}$ of non-decreasing functions z_n, is nonincreasing and bounded below by zero function. Hence, it converges uniformly on J_0 and the function $z(t)$ is the solution of (2.10) with $h(t) = 0$. By hypothesis (H_4), $z(t) = 0$ and from (3.21) we deduce that the sequence $\{x^n\}$ converges uniformly on J. Thus, $x(t) \overset{\text{def}}{=} \lim_{n\to\infty} x^n(t)$, $t \in J$, is continuous and a solution of (2.1) on J.

To prove the uniqueness, we suppose that x and $\bar{x}$ are two solutions of (2.1) with the same $u \in C(J, X)$. Define a sequence $\{v_n\}$ inductively by

$v_1(t) = w(t)$, any solution of (2.10) with

$$h(t) = h(0) = \sup\{\|x_t - \bar{x}_t\|_0 : t \in J_0\}$$

and

$$v_{n+1}(t) = M \int_0^t p\left(s, v_n(s), \int_0^s q(s, \tau, v_n(\tau))\, d\tau\right) ds. \qquad \ldots(3.22)$$

Using (2.10) and (3.22) we get by induction

$$\|x_t - \bar{x}_t\|_0 \leqslant v_n(t), \; t \in J_0, n \in N. \qquad \ldots(3.23)$$

The sequence $\{v_n\}$ is nonincreasing and bounded below by zero function. Hence it converges uniformly on J_0 and the function $v(t) \overset{\text{def}}{=} \lim_{n\to\infty} v_n(t)$ is the solution of (2.10) with $h(t) = 0$. By virtue of the hypothesis (H_4), $v(t) = 0$, $t \in J_0$ and consequently from (3.23) we have $x(t) = \bar{x}(t)$, $t \in J_0$. This completes the proof of Theorem 1.

Let $\{x^n\}$ be the sequence of the successive approximations, with $x^1 = u + Fv$ as the first approximation, converging to x. Define $u^n = v - x^n$, $n \in N$. It is easy to see that $u^n \to u = v - x$ uniformaly on J. Using definitions of u^n, x^1, (2.22), (2.23) and (3.14) we obtain

$$\|u_t^1\|_0 = \|v_t - u_t - Fv_t\|_0 \leqslant h(t),$$

and

$$\|u_t^{n+1}\|_0 \leqslant \|v_r - u_t - Fv_t\|_0 + \|Fv_t - Fx_t^n\|_0$$

$$\leqslant h(t) + M\int_0^t p\left(s, \|u_s^n\|_0, \int_0^s q(s, \tau, \|u_\tau^n\|_0)\,d\tau\right) ds.$$

Now an application of Lemma 2 yields

$$\|u\|_0 = \|v_t - x_t\|_0 \leqslant w(t),\ t \in J_0.$$

This completes the proof of Theorem 2.

If we choose $v = \bar{x}$ then by Theorem 2 we get (2.25) and the proof of Corollary 1 is complete.

Let $\{x^n\}$ be the sequence of the successive approximations, with $x^1 = u$ as the first approximation, converging to x. Define $u^n = x^n - v$. Then, it follows that $u^n \to u = x - v$ uniformly on J. Using definitions of u^n, x^1, (2.3), (2.11), (2.12), (2.22) and (2.27) we have

$$\|u_t^1\|_0 = \|u_t - v_t\|_0$$

and

$$\|u_t^{n+1}\|_0 \leqslant \|u_t - v_t\|_0 + M\int_0^t \left\|f\left(s, x_s^n, \int_0^u k(s, \tau, x_\tau^n)\,d\tau\right)\right\| ds$$

$$\leqslant \|u_t - v_t +\|_0\, M\int_0^t p^*\left(s, \|u_s^n\|_0, \int_0^s q^*(s, \tau, \|u_\tau^n\|_0)\,d\tau\right)\| ds.$$

Now, applying Lemma 2 with $h(t) = \|u_t - v_t\|_0$, we obtain

$$\|u_t\|_0 = \|x_t - v_t\|_0 \leqslant z(t),\ t \in J_0,$$

and the proof of Theorem 3 in complete.

Let $\{x^n\}$ denote the sequence of the successive approximations, with $x^1 = u + F\bar{x}$ as the first approximation, converging to x. Define $u^n = x^n - \bar{x}$. Then $u^n \to u = x - \bar{x}$ uniformly on J. Using definitions of u^n, x^1, (2.4), (2.22), (2.29) and (3.14) we get

$$\|u_t^1\|_0 = 0$$

and

$$\begin{aligned}\|u_t^{n+1}\| &= \|u_t + Fx_t^n - F\bar{x}_t + F\bar{x}_t - \bar{x}_t\|_0 \\ &= \|Fx_t^n - F\bar{x}_t\|_0 \\ &\leqslant h(t) + M\int_0^t p\left(s, \|u_s^n\|_0, \int_0^s q(s, \tau\|u_\tau^n\|_0\, d\tau\right) ds.\end{aligned}$$

Now, using Lemma 2, we get (2.28) and the proof of Theorem 4 is complete.

Remark 3 : It is important to note that the results established in Theorems 1-4 for solutions of (2.1) hold also for mild solutions of (1.1) when u and U in (2.1) are considered as the solution of (2.5) with given $\phi \in C$ as the initial function and the fundamental solution of (2.5) respectively.

4. Proofs of Theorems 5-7

Let $\phi \in C$ be given. We observe that the hypotheses of Theorem 1 hold for k and f. Therefore the existence of the unique mild solution x of (1.1) on J is ensured.

Let $\bar{x}$ be the mild solution of (1.1) on J corresponding to $\bar{\phi} \in C$ as the initial mapping. If u and $\bar{u}$ are the solutions of (2.5) corresponding to ϕ and $\bar{\phi} \in C$, as the initial mappings respectively, then by using (2.6) and proceeding as in the proof of Theorem 1 we obtain

$$\|u_t - \bar{u}_t\|_0 \leqslant \|\phi - \bar{\phi}\|_0 + M\|\phi(0) - \bar{\phi}(0)\|. \qquad \ldots(4.1)$$

From (4.1) and corollary 1, we obtain

$$\|x_t - \bar{x}_t\|_0 \leqslant w(t) \leqslant w(T),\ t \in J_0 \qquad \ldots(4.2)$$

where $w(t)$ is the minimal solution of (2.30) with

$$w_0 = \|\phi - \bar{\phi}\|_0 + M\|\phi(0) - \bar{\phi}(0)\|.$$

It is easy to see that if $\bar{\phi} \to \phi$ in C, then $w_0 \to 0$ and, since the zero function is the only solution of (2.30) with $w_0 = 0$ then

$$w(T) \to 0 \text{ as } \bar{\phi} \to \phi \qquad \ldots(4.3)$$

(see Walter[20], Theorem VIII, p. 67). Therefore, from (4.2) and (4.3) we have $\bar{x}(t) \to x(t)$ uniformly on J as $\bar{\phi} \to \phi$. This completes the proof of Theorem 5.

Let x and $\bar{x}$ be the mild solutions of (1.1) corresponding to the same $\phi \in C$ and μ, $\bar{\mu} \in E$ respectively. We see that the hypotheses of Theorem 4 are satisfied and hence we have

$$\|x_t - \bar{x}_t\|_0 \leqslant w(t) \leqslant w(T), \; t \in J_0 \qquad \text{...(4.4)}$$

where $w(t)$ is the minimal solution of (2.10) with $h(t)$ given by (2.29). Since the set $\{(s, \bar{x}_s, \int_0^s k(s, \tau, \bar{x}_\tau)\, d\tau, \mu) : 0 \leqslant s \leqslant T\}$ is compact in the product space $J_0 \times C \times X \times E$ and since f is continuous, it is easy to verify by elementary analysis (see Lang[11], p. 34) that $h(t)$ given by (2.29), tends to zero uniformly on J_0 as $\bar{\mu} \to \mu$ in E. Then it follows that

$$w(T) \to 0 \text{ as } \bar{\mu} \to \mu. \qquad \text{...(4.5)}$$

From (4.4) and (4.5), we obtain $\bar{x}(t) \to x(t)$ uniformly on J_0 as $\bar{\mu} \to \mu$. This completes the proof of Theorem 6.

By an application of Theorem 5, eqn. (1.1) has a unique mild solution x on J. By virtue of Lemmas 3.5.1 and 3.5.2 of Ladas and Lakshmikantham[9] with $u_0 = \phi(0)$ and $f(t) = f(t, x_t, \int_0^t k(t, s, x_s)\, ds)$, we can verify that x is also a strict solution of (1.1) on J. Now, since x is a strict solution of (1.1), the mapping $t \to f(t, x_t, \int_0^t k(t, s, x_s)\, ds)$ is continuous and so that x is also a mild solution of (1.1). (Lakshmikantham and Leela[10], Vol. II, p. 250). Therefore the uniqueness of a strict solution x follows from Theorem 5. This completes the proof of Theorem 7.

5. Proof of Theorems 8-10

The solution of eqn. (1.1) on $[-r, \infty)$ is given by

$$x(t) = U(t^+, 0)\, \phi(t^-) + \int_0^t U(t^+, s)\, f(s, x_s, \int_0^s k(s, \tau, x_\tau)\, d\tau)\, ds \qquad \text{...(5.1)}$$

where $t^+ = \max\{t, 0\}$ and $t^- = \min\{t, 0\}$. If $t \geqslant 0$, then from (5.1) and using (H_6), (2.13), (2.14) we obtain

$$\|x(t)\| \leqslant M\|\phi\|_0 + \int_0^t ML_2(s)\, \|x_s\|_0\, ds +$$

(equation continued on p. 742)

$$+ \int_0^t ML_2(s) \int_0^s L_1(\tau) \| x_\tau \|_0 \, d\tau \, ds. \qquad \ldots(5.2)$$

From (5.2) and by proceeding as in the proof of Theorem 1 we get

$$\|x_t\|_0 \leqslant M \|\phi\|_0 + \int_0^t ML_2(s) \|x_s\|_0 \, ds$$

$$+ \int_0^t ML_2(s) \int_0^s L_1(\tau) \|x_\tau\|_0 \, d\tau \, ds. \qquad \ldots(5.3)$$

By an application of Lemma 1 we get from (5.3),

$$\|x_t\|_0 \leqslant M \|\phi\|_0 [1 + \int_0^t ML_2(s) \exp (\int_0^s \{ML_2(\tau) + L_1(\tau)\} d\tau) \, ds]. \quad \ldots(5.4)$$

The boundedness of the solution $x(t)$ of (1.1) on R^+ follows from (5.4) and (2.15). Consequently, the solution $x(t)$ of (1.1) is bounded on $[-r, \infty)$ and proof of the Theorem 8 is complete.

Remark 4 : We note that our result established in Theorem 8 also yield the stability of a solution $x(t)$ of (1.1) if $\|\phi\|_0$ is small enough.

If $t \geqslant 0$ then using (5.1) and (2.16) — (2.18), we get

$$e^{\beta t} \|x(t)\| \leqslant M \|\phi\|_0 + \int_0^t ML_2(s) e^{\beta \tau} \|x_\tau\|_0 \, ds$$

$$+ \int_0^t ML_2(s) \int_0^s L_1(\tau) e^{\beta \tau} \|x_\tau\|_0 \, d\tau \, ds. \qquad \ldots(5.5)$$

From (5.5) and by proceeding as in the proof of Theorem 1 we obtain

$$e^{\beta t} \|x_t\|_0 \leqslant Me^{\beta r} \|\phi\|_0 + \int_0^t ML_2(s) e^{\beta s} \|x_s\|_0 \, ds$$

$$+ \int_0^t ML_2(s) \int_0^s L_1(\tau) e^{\beta \tau} \|x_\tau\|_0 \, d\tau \, ds. \qquad \ldots(5.6)$$

Applying Lemma 1 with $c(t) = e^{\beta t} \|x_t\|_0$, the inequality (5.6) yields

$$\|x_t\|_0 \leqslant Me^{\beta r} \|\phi\|_0 [1 + \int_0^t ML_2(s)$$

$$\exp [\int_0^s \{ML_2(\tau)\} + L_1(\tau)\} d\tau] \, ds] \, e^{-\beta t}. \qquad \ldots(5.7)$$

From (5.7) and (2.15), it follows that

$$\|x_t\|_0 \leqslant Q\,\|\phi\|_0\, e^{-\beta t},\ t \geqslant 0$$

where $Q > 0$ is a constant. This proves that the solution $x(t)$ of (1.1) is exponentially asymptotically stable and proof of the Theorem 9 is complete.

If $t \geqslant 0$ then using (5.1) and (2.19) — (2.21), we get

$$e^{-\beta t}\,\|x(t)\| \leqslant M\,\|\phi\|_0 + \int_0^t ML_2(s)\, e^{-\beta s}\,\|x_s\|_0\, ds$$

$$+ \int_0^t ML_2(s) \int_0^s L_1(\tau)\, e^{-\beta\tau}\,\|x_\tau\|_0\, d\tau\, ds. \qquad \ldots(5.8)$$

From (5.8) and by proceeding as in the proof of Theorem 1 we obtain

$$e^{-\beta t}\,\|x_t\|_0 \leqslant M\,\|\phi\|_0 + \int_0^t ML_2(s)\, e^{-\beta s}\,\|x_s\|_0\, ds$$

$$+ \int_0^t ML_2(s) \int_0^s L_1(\tau)\, e^{-\beta\tau}\,\|x_\tau\|_0\, d\tau\, ds. \qquad \ldots(5.9)$$

Applying Lemma 1 with $c(t) = e^{-\beta t}\,\|x_t\|_0$, the inequality (5.9) yields

$$\|x_t\|_0 \leqslant M\,\|\phi\|_0\,[1 + \int_0^t ML_2(s)$$

$$\times \exp\,[\int_0^s \{ML_2(\tau) + L_1(\tau)\}\, d\tau]\, ds]\, e^{\beta t}. \qquad \ldots(5.10)$$

From (5.10) and (2.15), it follows that

$$\|x_t\|_0 \leqslant Q\,\|\phi\|_0\, e^{\beta t},\ t > 0$$

where $Q > 0$ is a constant. This proves that the solution $x(t)$ of (1.1) is uniformly slowly growing and proof of the Theorem 10 is complete.

6. Applications

In order to illustrate the applications of some of our theorems established in previous sections, we consider the following partial functional integrodifferential equation

$$\frac{\partial z(x,t)}{\partial t} - (a(x,t)\ z_x(x,t))_x$$

$$= G(t, z(x, t-r), \int_0^t H(t, s, z(x, s-r))\, ds),$$

$$0 \leqslant x \leqslant 1,\ t \in J_0 \qquad \ldots(6.1)$$

with initial-boundary conditions

$$z(0, t) = z(1, t) = 0, \; t \in J_0 \qquad \ldots(6.2)$$

$$z(x, t) = \phi(x, t), \; 0 \leqslant x \leqslant 1, \; -r \leqslant t \leqslant 0. \qquad \ldots(6.3)$$

The functions a, H and G in (6.1) satisfy the following conditions :

(C_1) $a(x, t)$ is a positive continuous function defined on $[0, 1] \times J_0$ having a continuous first partial derivative $a_x(x, t)$ on $[0, 1] \times J_0$ and satisfy

$$| a(x, t) - a(x, s) | \leqslant K | t - s |^{\eta} \qquad \ldots(6.4)$$

$$| a_x(x, t) - a_x(x, s) | \leqslant K | t - s |^{\eta} \qquad \ldots(6.5)$$

for all $s, t \in J_0$, $x \in [0, 1]$, and $0 < \eta < 1$

(C_2) $H : J_0 \times J_0 \times R \to R$ is continuous and satisfies

$$| H(t, s, v) - H(t, s, \bar{v}) | \leqslant q(t, s, | v - \bar{v} |) \qquad \ldots(6.6)$$

for $0 \leqslant s \leqslant t \leqslant T$ and $v, \bar{v} \in R$ where q is defined as in Hypothesis (H_1)

(C_3) $G : J_0 \times R \times R \to R$ is continuous and satisfies

$$| G(t, v_1, v_2) - G(t, \bar{v}_1, \bar{v}_2) | \leqslant p(t, | v_1 - \bar{v}_1 |, | v_2 - \bar{v}_2 |) \qquad \ldots(6.7)$$

for $0 \leqslant t \leqslant T$ and $v_1, v_2;\; \bar{v}_1, \bar{v}_2 \in R$ where p is defined as in Hypothesis (H_2).

(C_4) The functions q and p involved in (6.6) and (6.7) are such that the equation (2.30) has a solution on J_0 for each $w_0 \geqslant 0$ and $w(t) = 0$ is the only solution of (2.30) with $w_0 = 0$.

Let $X = C([0, 1], R)$ denote the Banach space with norm $\|y\| = \max_{0 \leqslant x \leqslant 1} | y(x) |$, $y \in X$. Define an operator $A : X \to X$ by

$$(A(t) u)(x) = -(a(x, t) z_x(x,t))_x$$

wiih domain $D(A)$ defined as $D(A) = \{u \in X, (a(., 0) z_x(., .))_x \in X, z(0, t) = z(1,t) = 0\}$. We now define mappings $k : J_0 \times J_0 \times C \to X$ and $f : J_0 \times C \times X \to X$ as follows

$$k(t, s, \phi)(x) = H(t, s, \phi(-r) x) \qquad \ldots(6.8)$$

$$f(t, \phi, y)(x) = G(t, \phi(-r)(x), y(x)). \qquad \ldots(6.9)$$

The problem (6.1) — (6.3) can be formulated abstractly as

$$\left.\begin{array}{l} u'(t) + A(t) u(t) = f\left(t, u_t, \int_0^t k(t, s, u_s)\, ds\right), \; t \in J_0 \\ u(t) = \phi(t), \; -r \leqslant t \leqslant 0. \end{array}\right\} \qquad \ldots(6.10)$$

In view of definitions (6.8) and (6.9), it is easy to observe from (C_2) and (C_3) that the Hypotheses (H_1) and H_2) hold. Now, a application of Theorem 7 yields a unique strict solution $z(x, t) = u(\phi)(t)x$, $x \in [0, 1]$, $t \in J$ of (6.1) — (6.3).

For the following discussion, we suppose that the solution $u(t)$ of (6.10) exists on $[-r, \infty)$.

Suppose that the functions $H : R^+ \times R^+ \times R \to R$ and $G : R^+ \times R \times R \to R$ in (6.1) satisfy the following conditions:

(C_5) For $t, s \in R^+$, $v, v_1, v_2 \in R$

$$| H(t, s, v) | \leqslant L_1 e^{\beta(t-s)} | v | \qquad \ldots(6.11)$$

$$| G(t, v_1, v_2) | \leqslant L_2 e^{-\beta t} [| v_1 | + | v_2 |] \qquad \ldots(6.12)$$

where L_1 and L_2 are nonnegative constants. Define mappings $k : R^+ \times R^+ \times C \to X$ and $f : R^+ \times C \times X \to X$ as in (6.8) and (6.9) respectively. From (6.8), (6.9), (6.11) and (6.12), it is easy to observe that the hypothesis (H_9) hold.

If the fundamental solution $U(t, s)$ $t, s \in R^+$, of eqn. (2.5) satisfies condition (2.16) then by an application of Theorem 9, the solutions $z(x, t)$ of (6.1)–(6.3) tends to zero as $t \to \infty$.

Theorems 8 and 10 can be applied equally well to study the boundedness and the growth of the solutions of (6.1)–(6.3) by using suitable conditions on the functions H and G involved in (6.1) and on the fundamental solution U of eqn. (2.5). We omit the details.

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ON EXTENSION OF MAPS IN TOPOLOGICAL SPACES

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The purpose of this paper is to investigate the piecewise definition of certain maps namely irresolute maps, semihomeomorphisms, etc.

1. INTRODUCTION

Levine[5] introduced the notion of semiopen sets in topological spaces, to study the properties of functions weaker than continuous functions. The purpose of this paper is to investigate the piecewise definition of certain maps. In this direction, we introduce in section 2 the notion of strongly *nbd*-finite family of sets and study its properties and applications In this paper, X and Y always denote topological spaces.

2. STRONGLY *nbd*-FINITE

Definition 2.1—A family $\{A_m;\ m \in M\}$ of subsets of X is said to be strongly *nbd*-finite if for each x in X, there is an open set V containing x, satisfying one of the following conditions :

(a) $V \cap A_m = \phi$ for every m in M.

(b) There is a non-empty finite subset N of M such that

i) $V \cap A_m \neq \phi$ for every m in N,

ii) $V \cap A_m \subset A_k$ for every m, k with m in N and k in N, and

iii) $V \cap A_m = \phi$ for every $m \notin N$.

Every strongly *nbd*-finite family is *nbd*-finite[4]. However, a *nbd*-finite family need not be strongly *nbd*-finite, as shown by the following example.

Example 2.2—Let $X = \{a, b, c\}$ and

$T = \{\phi, \{a\}, \{b, c\}, X\}$. Then $\{\{b\}, \{c\}\}$

is *nbd*-finite but not strongly *nbd*-finite.

Before giving the properties we consider the following.

It is easy to prove the following results that are improved results of Theorem 1.8 of Crossely and Hildebrand[2].

Lemma 2.3—If A and B are subsets of X such that A is open, then $A \cap B = \phi$ implies $A \cap \text{cl}(B) = \phi$ where cl (B) denotes the closure of B in X.

Corollary 2.4—Under the hypothesis of Lemma 2.3, $A \cap B = \phi$ implies $A \cap \text{scl}(B) = \phi$ where scl (B) is the semiclosure[2] of B in X.

Lemma 2.5—If A and B are subsets of X such that A is an α-set[6] (Alpha set), then $A \cap B = \phi$ implies $A \cap \text{scl}(B) = \phi$.

3. Properties

Theorem 3.1—If $\{A_m;\ m \in M\}$ is strongly *nbd*-finite, then $\{\text{scl}(A_m);\ m \in M\}$ is also strongly *nbd*-finite.

Proof : Let x be in X. Choose V as in Definition 2.1. If 2.1 (a) holds, by Corollary 2.4, $V \cap \text{scl}(A_m) = \phi$ for every m in M. So, let us assume that 2.1 (b) holds. It is enough to prove that $V \cap \text{scl}(A_m) \subset \text{scl}(A_k)$ where $m \in N$ and $k \in N$.

Let $y \in V \cap \text{scl}(A_m)$, $m \in N$. Let S be any semiopen set containing y. Then, by Theorem 1.9 of Crossely and Hildebrand[2], $V \cap S$ is semiopen. Hence, by Lemma 3 of Noiri[9], $V \cap S \cap A_m \neq \phi$. If $k \in N$, then by 2.1 (b) (ii), $V \cap S \cap A_k \neq \phi$ which implies $S \cap A_k \neq \phi$ and hence $y \in \text{scl}(A_k)$. This completes the proof.

Theorem 3.2—Let $\{A_m;\ m \in M\}$ be *nbd*-finite in X. Then $\{\text{scl}(A_m);\ m \in M\}$ is also *nbd*-finite.

Proof : Analogous to Theorem 9.2 of Dugundji[4].

In Dugundji[4], it has been proved that union of closed sets of a *nbd*-finite family is closed. The following is an analog of this result for semiclosed[2] sets.

Theorem 3.3—If $\{A_m;\ m \in M\}$ is a strongly *nbd*-finite family of semiclosed sets in X, then $B = \bigcup_{m \in M} A_m$ is semiclosed.

Proof : Let x be in $X - B$. Choose V as in Definition 2.1. If 2.1 (a) holds, take $S = V$. If 2.1 (b) holds, take $S = \bigcup_{m \in N} [V \cap (X - A_m)]$. Since union of semiopen sets is semiopen, S is semiopen. Clearly, $x \in S$. It is easy, by computation, to prove that $B \cap S = \phi$ so that $x \in S \subset X - B$ which implies that $X - B$ is semiopen and hence B is semiclosed.

Remark 3.4 : Theorem 3.3 need not be true for a *nbd*-finite family.

Remark 3.5 : If $\{A_m;\ m \in M\}$ is strongly *nbd*-finite, then $\bigcup_{m \in M} \{\text{scl}(A_m)\}$ is semiclosed.

The following simple result on covering will be utilised in sequel.

Theorem 3.6—Let $\{A_m;\ m \in M\}$ be an α-set covering of X and B, a subset of X.

Then B is semiopen (resp. semiclosed) in X if and only if $B \cap A_m$ is semiopen (resp. semiclosed) in the subspace A_m for every m.

PROOF : *Necessary part*—Let B be semiopen in X. Then, by Proposition 1 of Njastad[6], $B \cap A_m$ is semiopen in X. By Theorem 1 of Noiri[7], $B \cap A_m$ is semiopen in A_m.

Now, let B be semiclosed in X. Then $X - B$ is semiopen in X. Again, by Proposition 1 of Njastad[6], $(X - B) \cap A_m$ is semiopen in X and hence semiopen in A_m. As $(X - B) \cap A_m = A_m (B \cap A_m)$, $B \cap A_m$ is semiclosed in A_m.

Sufficient part—Assume that $B \cap A_m$ is semiopen in A_m for every m. Then $B \cap A_m$ is semiopen in X for every m. Since $B = B \cap X = B \cap (\bigcup_{m \in M} A_m) = \bigcup_{m \in M} (B \cap A_m)$, B is semiopen in X. Assume that $B \cap A_m$ is semiclosed in A_m for each m. Then $A_m - (B \cap A_m)$ is semiopen in A_m and hence semiopen in X, by Theorem 1 of Noiri[7]. It is easy to see that

$X - B = \bigcup_{m \in M} (A_m - (B \cap A_m))$ which implies that B is semiclosed in X. This completes the proof.

The main property of a strongly *nbd*-finite family is the next theorem.

Theorem 3.7—Let $\{A_m; m \in M\}$ be a covering of X such that all A_m are semiclosed and form a strongly *nbd*-finite family. Let B be a subset of X. If $B \cap A_m$ is semiclosed (resp. semiopen) in A_m for every m, then B is semiclosed (resp. semiopen) in X.

PROOF : If $B \cap A_m$ is semiclosed in A_m for every m, then by Lemma 2.19 of Sivaraj[11], $B \cap A_m$ is semiclosed in X and as $\{B \cap A_m; m \in M\}$ is strongly *nbd*-finite, $\bigcup_m (B \cap A_m)$ is semiclosed in X which implies B is semiclosed in X.

If $B \cap A_m$ is semiopen in A_m, it can be proved, by considering complements, that B is semiopen in X. This completes the proof.

4. APPLICATIONS

In analysis, continuous functions are frequently defined piecewise. Here we have an analog situation for irresolute[3] and semi continuous[5] maps.

Theorem 4.1—Let $\{A_m; m \in M\}$ be a covering of α-sets or a strongly *nbd*-finite covering of semiclosed sets of X. For each m, let f_m be a given irresolute (resp. semi continuous) map of A_m into Y. Assume that f_m and f_k coincide on $A_m \cap A_k$ for every m in M and for every k in M. Then there is a unique irresolute (resp. semi continuous) map f of X into Y which extends each f_m.

PROOF : Existence and uniquencss of a map f follow from Theorem 6.7 (p.13) of Dugundji[4].

To prove that f is irresolute (resp. semi continuous), let B be a semiopen (resp. open) subset of Y. Since f_m is irresolute (resp. semi continuous), $f_m^{-1}(B)$ is semiopen in A_m for every m. Since $f^{-1}(B) \cap A_m = f_m^{-1}(B)$, by Theorems 3.6 and 3.7, $f^{-1}(B)$ is semiopen in X. This completes the proof.

Theorem 4.2—Let $\{B_m;\ m \in M\}$ be an α-set or a strongly *nbd*-finite semiclosed covering of Y. Let f be a map of X into Y. If the restriction f_m of f to $f^{-1}(B_m)$ is a semiopen[1] (resp. semiclosed[5]) map of $f^{-1}(B_m)$ into B_m for each m in M, then f is semiopen (resp. semiclosed).

PROOF : Let A be an open (resp. closed) subset of X. Then $A \cap f^{-1}(B_m)$ is open (resp. closed) in $f^{-1}(B_m)$. Since $f_m(A \cap f^{-1}(B_m)) = f(A) \cap B_m$ and since f_m is semiopen (resp. semiclosed), $f(A) \cap B_m$ is semiopen (resp. semiclosed) in B_m. Then the proof follows from Theorems 3.6 and 3.7.

Theorem 4.3—Let $\{B_m;\ m \in M\}$ be a covering of α-sets of Y. Let f be a map of X into Y such that $\{f^{-1}(B_m);\ m \in M\}$ is a covering of α-set of X. If the restriction f_m of f to $f^{-1}(B_m)$ is a presemiopen[3] (resp. presemiclosed[10]) map of $f^{-1}(B_m)$ into B_m for each m in M, then f is presemiopen (resp. presemiclosed).

PROOF : Analogous to Theorem 4.2.

Theorem 4.4—Let $\{B_m;\ m \in M\}$ be an α-set covering of Y and let f be an injective map of X onto Y such that $\{f^{-1}(B_m);\ m \in M\}$ is an α-set covering of X. If the restriction f_m of f to $f^{-1}(B_m)$ is a semihomeomorphism[3] of $f^{-1}(B_m)$ to B_m for each m in M, then f is a semihomeomorphism.

PROOF : Follows from Theorems 4.1 and 4.3.

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ON MINIMAL PAIRWISE HAUSDORFF BITOPOLOGICAL SPACES

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The purpose of this paper is to investigate minimal pairwise Hausdorff bitopological spaces. The pairwise semiregularization of a bitopological space is introduced and a brief outline of its properties is given. A new concept of pairwise almost compactness is defined and filter characterizations of (pairwise Hausdorff) pairwise almost compact spaces are obtained. The above concepts are used to prove bitopological analogues of two well-known Theorems given by M. Katětov and B. Banaschewski for minimal Hausdorff topological spaces.

INTRODUCTION

The concept of minimal pairwise Hausdorff bitopological spaces was initiated by Raghavan and Reilly[8]. The purpose of this paper is to continue the study of the above concept.

In section 1 we introduce the pairwise semiregularization of a bitopological space and examine its properties. We investigate also a certain type of bitopological adherence and convergence of filters. In section 2 we introduce a new concept of pairwise almost compactness and discuss its relationship to other well-known forms of bitopological compactness. In section 3 we give filter characterizations of (pairwise Hausdorff) pairwise almost compact spaces. In section 4 we improve a characterization of minimal pairwise Hausdorff spaces given by Raghavan and Reilly, and generalize two well-known Theorems of Katětov[5] and Banaschewski[1]. Finally, we prove that the concept of minimal pairwise Hausdorff spaces is not product invariant.

Throughout this paper if (X, τ_1, τ_2) is a bitopological space the τ_i-interior of a set $A \subset X$ is denoted by $<A>_i$ or $<A>_{\tau_i}$ and the τ_i-closure of A by $[A]_i$ or $[A]_{\tau_i}$, where $i = 1, 2$. The set of all τ_i-neighbourhoods of a point $x \in X$ is denoted by $\mathcal{N}_i(x)$ and the set of all τ_i-open neighbourhoods of x by $\tau_i(x)$, $i = 1, 2$. Whenever we deal with a statement involving the topologies τ_i and τ_j it will be understood that $i, j = 1, 2$ and $i \neq j$. Generally terms and notations not explained in this paper are those of Kelly[6], Cooke and Reilly[3] and Raghavan and Reilly[8].

1. PRELIMINARY DEFINITIONS AND THEOREMS

A. *Pairwise Semiregularization*

Let (X, τ_1, τ_2) be a bitopological space. According to Singal and Singal[10] a subset A of X is said to be (i, j)-regularly open if $A = <[A]_j>_i$.

Definition 1.1—(X, τ_1, τ_2) is said to be (i, j)-semiregular if for each $x \in X$ the collection of all (i, j)-regularly open neighbourhoods of x is a τ_i-open neighbourhood base at x. If (X, τ_1, τ_2) is $(1, 2)$-and $(2, 1)$-semiregular it is called pairwise semiregular (henceforth abbreviated as p-semiregular).

Since the intersection of two (i, j)-regularly open sets is an (i, j)-regularly open set, we obtain the following result.

Proposition 1.2—The collection of all (i, j)-regularly open sets forms a base for a topology τ_i^s on X, coarser than τ_i.

Definition 1.3—The bitopological space (X, τ_1^s, τ_2^s) is called the pairwise semiregularization (henceforth abbreviated as p-semiregularization) of (X, τ_1, τ_2).

Theorem 1.4—A bitopological space (X, τ_1, τ_2) is (i, j)-semiregular (resp. p-semiregular) iff $\tau_i^s = \tau_i$ (resp. $\tau_1^s = \tau_1$ and $\tau_2^s = \tau_2$).

PROOF : Follows from Definition 1.1 and Proposition 1.2.

Using the fact that the set $<[A]_j>_i$ is (i, j)-regularly open for each $A \subset X$, we can easily prove the following lemma.

Lemma 1.5—If (X, τ_1^s, τ_2^s) is the p-semiregularization of (X, τ_1, τ_2), then $[V]_{\tau_j} = [V]_{\tau_j^s}$ and $<[V]_{\tau_j}>_{\tau_i} = <[V]_{\tau_j^s}>_{\tau_i^s}$ for each $V \in \tau_i$.

Proposition 1.6—A subset A of X is (i, j)-regularly open in (X, τ_1, τ_2) iff it is (i, j)-regularly open in (X, τ_1^s, τ_2^s).

PROOF : Follows from Lemma 1.5.

Theorem 1.7—The p-regularization (X, τ_1^s, τ_2^s) of (X, τ_1, τ_2) is a p-semiregular bitopological space.

PROOF : Follows from Propositions 1.2, 1.6 and Theorem 1.4.

Our last result is easily proved.

Theorem 1.8—(X, τ_1, τ_2) is p-Hausdorff[3] iff (X, τ_1^s, τ_2^s) is p-Hausdorff.

B. *θ_{ij}-adherence and convergence of filters*

Let (X, τ_1, τ_2) be a bitopological space.

Definition 1.9—A point $x \in X$ is said to be θ_{ij}-adherent point of $A \subset X$ if $[V]_j \cap A \neq \phi$ for each $V \in \mathfrak{N}_i(x)$. The set of all θ_{ij}-adherent points of A is denoted by $[A]_{ij}$.

Definition 1.10—A point $x \in X$ is said to be θ_{ij}-limit point (resp. θ_{ij}-adherent point) of a filter $\mathscr{F}$ on X if $\mathscr{F} \supset \{[V]_j : V \in \mathscr{N}_i(x)\}$ (resp. if $x \in \cap \{[F]_{ij} : F \in \mathscr{F}\}$). If $\mathscr{B}$ is a filterbase on X, then a point $x \in X$ is called θ_{ij}-limit point (resp. θ_{ij}-adherent point) of $\mathscr{B}$ if x is θ_{ij}-limit point (resp. θ_{ij}-adherent point) of the filter $\mathscr{F}$ generated by $\mathscr{B}$. The set of all θ_{ij}-limit points of a filter $\mathscr{F}$ (resp. filterbase $\mathscr{B}$) is denoted by θ_{ij}-lim $\mathscr{F}$ (resp. θ_{ij}-lim $\mathscr{B}$) and the set of all θ_{ij}-adherent points of a filter $\mathscr{F}$ (resp. filterbase $\mathscr{B}$) by θ_{ij}-adh $\mathscr{F}$ (resp. θ_{ij}-adh $\mathscr{B}$).

It is obvious that a point $x \in X$ is θ_{ij}-limit point (resp. θ_{ij}-adherent point) of a filter base $\mathscr{B}$ on X iff for each $V \in \mathscr{N}_i(x)$ there exists a $B \in \mathscr{B}$ such that $B \subset [V]_j$ (resp. iff $x \in \cap \{[B]_{ij} : B \in \mathscr{B}\}$).

We recall that a filter $\mathscr{F}$ on X is called τ_i-open if it has a base $\mathscr{B}$ consisting exclusively by τ_i-open sets.

The proofs of the following results are straightforward and therefore they are omitted.

Proposition 1.11—(a) For each $A \subset X$, $[A]_i \subset [A]_{ij}$.

(b) If $A \subset B \subset X$, then $[A]_{ij} \subset [B]_{ij}$.

(c) If $A \in \tau_j$, then $[A]_i = [A]_{ij}$.

Proposition 1.12—(a) If $\mathscr{F}$ is a filter on X, then θ_{ij}-lim $\mathscr{F} \subset \theta_{ij}$-adh $\mathscr{F}$, τ_i-lim $\mathscr{F} \subset \theta_{ij}$-lim $\mathscr{F}$ and τ_i-adh $\mathscr{F} \subset \theta_{ij}$-adh $\mathscr{F}$.

(b) If $\mathscr{F}$, $\mathscr{F}^*$ are two filters on X such that $\mathscr{F} \subset \mathscr{F}^*$, then θ_{ij}-lim $\mathscr{F} \subset \theta_{ij}$-lim $\mathscr{F}^*$ and θ_{ij}-adh $\mathscr{F}^* \subset \theta_{ij}$-adh $\mathscr{F}$.

(c) If $\mathscr{F}$ is a τ_j-open filter on X, the θ_{ij}-adh $\mathscr{F} = \tau_i$-adh $\mathscr{F}$.
Finally, we prove two useful lemmas.

Lemma 1.13—If (X, τ_1, τ_2) is (i, j)-semiregular, then θ_{ij}-lim $\mathscr{B} = \tau_i$-lim $\mathscr{B}$ for each τ_i-open filterbase $\mathscr{B}$ on X.

PROOF : By Proposition 1.12 (x), τ_i-lim $\mathscr{B} \subset \theta_{ij}$-lim $\mathscr{B}$.

Let now x be a θ_{ij}-limit point of $\mathscr{B}$ and V a τ_i-open neighbourhood of x. Since (X, τ_1, τ_2) is (i, j)-semiregular there exists an (i, j)-regularly open neighbourhood W of x such that $x \in W \subset V$. By the hypothesis $x \in \theta_{ij}$-lim $\mathscr{B}$, so there exists a $B \in \mathscr{B}$ with $B \subset [W]_j$. Finally, since $B \in \tau_i$ and $W = <[W]_j>_i$ it is clear that $B \subset W \subset V$. Thus $x \in \tau_i$-lim $\mathscr{B}$ and the proof is complete.

Lemma 1.14—If (X, τ_1, τ_2) is a bitopological space, then for each filter $\mathscr{F}$ on X there exists a τ_j-open filter $\mathscr{F}^*$ on X, coarser then $\mathscr{F}$, such that τ_i-adh $\mathscr{F}^* = \theta_{ij}$-adh $\mathscr{F}$.

PROOF : Let $\mathscr{F}$ be a filter on X. If θ_i-adh $\mathscr{F} = X$, then the τ_j-open filter $\mathscr{F}^* = \{X\}$ is coarser than $\mathscr{F}$ and τ_i-adh $\mathscr{F}^* = X$. If θ_{ij}-adh $\mathscr{F} \neq X$, then for

each $x \in X-(\theta_{ij}\text{-adh } \mathscr{F})$ there exist a $V_x \in \tau_i(x)$ and an $F_x \in \mathscr{F}$ such that $F_x \cap [V_x]_j = \phi$. If now $\mathscr{V}$ is the collection of all these V_x's it is easily proved that $\mathscr{B} = \{X - [V]_j : V \in \mathscr{V}\}$ is a τ_j-open filterbase which generates a filter $\mathscr{F}^*$ on X, coarser than $\mathscr{F}$. Since $\mathscr{V}$ is a τ_i-open cover of $X-(\theta_{ij}\text{-adh } \mathscr{F})$, it is clear that

$$\tau_i\text{-adh } \mathscr{F}^* = \cap \{[X-[V]_j]_i : V \in \mathscr{V}\} = X - \cup \{<[V]_j>_i : V \in \mathscr{V}\} \subset \theta_{ij}\text{-adh } \mathscr{F}.$$

Finally, by Proposition 1.12, $\theta_{ij}\text{-adh } \mathscr{F} \subset \tau_i\text{-adh } \mathscr{F}^*$ and hence the proof is complete.

2. Pairwise Almost Compactness

Definition 2.1—A bitopological space (X, τ_1, τ_2) is called (i, j)-almost compact (henceforth abbreviated as $(i, j) - a - c$) if given a point $c \in X$, a τ_i-open cover $\mathscr{G} = \{G_k : k \in K\}$ of $X - \{c\}$ and a τ_j-open neighbourhood V of c, there exists a finite subcollection $\{G_{k_m} : m = 1, 2,\ldots, n\}$ of $\mathscr{G}$ with $X = [V]_i \cup (\cup \{[G_{k_m}]_j : m = 1, 2,\ldots, n\})$. If (X, τ_1, τ_2) is (1, 2)-and $(2, 1)-a-c$, then it is called pairwise almost compact (henceforth abbreviated as $p - a - c$).

The following result is an immediate consequence of the above definition and Lemma 1.5.

Theorem 2.2—If (X, τ_1, τ_2) is $p-a-c$, then the p-semiregularization (X, τ_1^s, τ_2^s) of (X, τ_1, τ_2) is also $p-a-c$.

It is clear from the definitions and Example 2.3 below that p-a-compactness is strictly weaker than semi-compactness and p-compactness (Cooke and Reilly[3], Swart[11] and Fletcher *et al.*[4])

Example 2.3—Let $X = [0, 1]$. The collections

$$\mathscr{B}_1 = \{X\} \cup \{(a, b) : 0 \leqq a < b \leqq 1\} \cup \{(a, 1] : 0 \leqq a < 1\} \cup \left\{[0, b) - \left\{\frac{1}{n} : n \in \mathbb{N}\right\} : 0 < b \leqq 1\right\}$$

$$\mathscr{B}_2 = \{X\} \cup \{(a, b) : 0 \leqq a < b \leqq 1\} \cup \{[0, b) : 0 < b \leqq 1\} \cup \left\{(a, 1] - \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} : 0 \leqq a < 1\right\}$$

are bases for the topologies τ_1 and τ_2 respectively.

The collection

$$\mathscr{A} = \left\{[0, 1) - \left\{\frac{1}{n} : n \in \mathbb{N}\right\}, \left(\frac{1}{2}, 1\right]\right\} \cup \left\{\left(\frac{1}{n+1}, \frac{1}{n-1}\right) : n \in \mathbb{N} - \{1\}\right\}$$

is clearly a $\tau_1\tau_2$-open and also a pairwise open cover of X. Since any finite sub-collection of $\mathscr{H}$ does not cover X, (X, τ_1, τ_2) is neither semi-compact nor p-compact. However, since the topological space (X, τ), where τ is the usual topology on X, is compact and $[B]_{\tau_j} = [B]_\tau$ for each $B \in \mathscr{B}_i$, it is easily proved that (X, τ_1, τ_2) is $p-a-c$.

By the following examples we show that B-compactness (Cooke and Reilly[3] and Birsan[2]) does not imply and is not implied by p-a-compactness.

Example 2.4—Let $X = [0, 1]$, $\tau_1 = \{\phi, X, \{0\}\} \cup \{[0, \alpha) : 0 < \alpha \leqq 1\}$ and $\tau_2 = \{\phi, X, \{1\}\} \cup \{(a, 1] : 0 \leqq \alpha < 1\}$. The bitopological space (X, τ_1, τ_2) is clearly B-compact. However, since there exist a τ_1-open cover $\mathscr{H} = \left\{\left[0, 1 - \frac{1}{n}\right) : n \in \mathbb{N} - \{1\}\right\}$ of $X - \{1\}$ and a τ_2-open neighbourhood $V = \{1\}$ of the point $c = 1$ such that for each finite subset $\{m_1, m_2, \ldots, m_n\}$ of $\mathbb{N} - \{1\}$, $[V]_1 \cup \left(U \left\{\left[\left[0, 1 - \frac{1}{m_k}\right)\right]_2 : k = 1, 2, \ldots, n\right\}\right) = \{1\} \cup \left[0, 1 - \frac{1}{m_0}\right] \neq X$ $(m_0 = \max\{m_1, m_2, \ldots, m_n\})$, (X, τ_1, τ_2) is not $p - a - c$.

Example 2.5—Let (X, τ_1, τ_2) be the $p - a - c$ space of Example 2.3. Since the topological spaces (X, τ_1) and (X, τ_2) are not compact, it is clear that (X, τ_1, τ_2) is not B-compact.

Finally, by the next two examples, we show that p-a-compactness does not imply and is not implied by Mukherjee's[7] p-a-compactness (henceforth denoted by M-a-compactness).

Example 2.6—The topological space (X, τ_1, τ_2), where $X = \mathbb{R}$, τ_1 the discrete topology and τ_2 the topology of finite complements on $\mathbb{R}$, is clearly $p - a - c$ but it is not $M - a - c$.

Example 2.7—The bitopological space (X, τ_1, τ_2) of Example 2.4 is not $p-a-c$. However, since (X, τ_1) and (X, τ_2) are compact topological spaces, (X, τ_1, τ_2) is $M-a-c$.

3. Filter Characterizations of p-a-Compactness

Theorem 3.1—Let (X, τ_1, τ_2) be a bitopological space. The following are equivalent.

(i) (X, τ_1, τ_2) is $(i, j)-a-c$.

(ii) For each point $c \in X$ and for each τ_j-open filterbase $\mathscr{B}$ on X with τ_i-adh $\mathscr{B} \subset \{c\}$, $c \in \theta_{ji}$-lim $\mathscr{B}$.

(iii) For each point $c \in X$ and for each filter $\mathscr{F}$ on X with θ_{ij}-adh $\mathscr{F} \subset \{c\}$, $c \in \theta_{ji}$-lim $\mathscr{F}$.

PROOF : (i) $\Rightarrow$ (ii) Let c be a point of X, $\mathscr{B}$ a τ_j-open filterbase on X with τ_i-adh $\mathscr{B} \subset \{c\}$ and V a τ_j-open neighbourhood of c. Since $\cap \{[B]_i : B \in \mathscr{B}\} \subset \{c\}$, the collection $\{X - [B]_i : B \in \mathscr{B}\}$ is a τ_i-open cover of $X - \{c\}$. Therefore there exists a finite subcollection $\{B_k : k = 1, 2, \ldots, n\}$ of $\mathscr{B}$ such that

$$X = [V]_i \cup (U\{[X - [B_k]_i]_j : k = 1, 2, \ldots, n\})$$

$$= [V]_i \cup (X - \cap \{<[B_k]_i>_j : k = 1, 2, \ldots, n\}).$$

Finally, if we choose a $B \in \mathscr{B}$ with $B \subset B_1 \cap B_2 \cap \ldots \cap B_n$ it is clear that $B \subset [V]_i$ and hence $c \in \theta_{ji}$-lim $\mathscr{B}$.

(ii) $\Rightarrow$ (i) Suppose (X, τ_1, τ_2) is not $(i, j)-a-c$. Then there exist a point $c \in X$, a τ_i-open cover $\{G_1 : l \in L\}$ of $X - \{c\}$ and a τ_j-open neighbourhood V of c, such that for each finite subset K of L, $X \neq [V]_i \cup (U\{[G_k]_j : k \in K\})$.

Since now $(X - [V]_i) \cap (\cap \{X - [G_k]_j : k \in K\}) \neq \phi$, it is easily proved that $\mathscr{B} = \{ \bigcap_{k \in K} (X - [G_k]_j) : K \subset L,\ K \text{ finite}\}$ is a τ_j-open filterbase on X with τ_i-adh $\mathscr{B} \subset \{c\}$ such that $B \not\subset [V]_i$ for each $B \in \mathscr{B}$. Therefore $c \notin \theta_{ji}$-lim $\mathscr{B}$ and the proof is complete.

(ii) $\Rightarrow$ (iii) Follows from Lemma 1.14 and Proposition 1.12 (b).

(iii) $\Rightarrow$ (ii) Follows from Proposition 1.12 (c).

Theorem 3.2—If (X, τ_1, τ_2) is p-Hausdorff, then the following are equivalent.

(i) (X, τ_1, τ_2) is $(i, j)-a-c$.

(ii) For each τ_j-open filterbase $\mathscr{B}$ on X with τ_i-adh $\mathscr{B} = \{c\}$, $c \in \theta_{ji}$-lim $\mathscr{B}$.

(iii) For each filter $\mathscr{F}$ on X with $\theta_{ij}-$adh $\mathscr{F} = \{c\}$, $c \in \theta_{ji}$-lim $\mathscr{F}$.

PROOF : (i) $\Rightarrow$ (ii) Follows from Theorem 3.1.

(ii) $\Rightarrow$ (i) We need only prove that for each τ_j-open filterbase $\mathscr{B}$ on X with τ_i-adh $\mathscr{B} = \phi$, θ_{ji}-lim $\mathscr{B} = X$. Let $\mathscr{B}$ be a τ_j-open filterbase on X with τ_i-adh $\mathscr{B} = \phi$ and c a point of X. It is easily proved that $\mathscr{B}^* = \{B \cup G : B \in \mathscr{B} \text{ and } G \in \tau_j(c)\}$ is a τ_j-open filterbase on X with τ_i-adh $\mathscr{B}^* = \{c\}$. So, by the hypothesis, $c \in \theta_{ji}$-lim $\mathscr{B}^*$. Since now the filter $\mathscr{F}^*$ generated by $\mathscr{B}^*$ is coarser than the filter $\mathscr{F}$ generated by $\mathscr{B}$, it is clear, by Proposition 1.12 (b), that $c \in \theta_{ji}$-lim $\mathscr{B}$ and the proof is complete.

(ii) $\Leftrightarrow$ (iii) Follows from Lemma 1.14 and Proposition 1.12.

By the following example one can see that the condition "p-Hausdorff" can not be omitted in Theorem 3.2.

Example 3.3—Let $(\mathbb{R}^2, \tau_1, \tau_2)$ be the bitopological product [Swart[11]], of the spaces $(\mathbb{R}, \tau_1', \tau_2')$ and $(\mathbb{R}, \tau_1'', \tau_2'')$, where τ_1' is the discrete topology, $\tau_2' = \tau_1''$ the usual topology and τ_2'' the topology of finite complements on $\mathbb{R}$. Clearly $(\mathbb{R}^2, \tau_1, \tau_2)$ is not p-Hausdorff. Since there exist a τ_1-open cover $\mathscr{H} = \{\{x\} \times \mathbb{R} : x \in \mathbb{R}\}$ of $\mathbb{R}^2$-$\{(0, 0)\}$ and a τ_2-open neighbourhood $V = (-1, 1) \times \mathbb{R}$ of the point $c = (0, 0)$ such that for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of $\mathbb{R}$

$$[V]_{\tau_1} \cup (\cup \{[\{x_k\} \times \mathbb{R}]_{\tau_2} : k = 1, 2, \ldots, n\})$$

$$= ((-1, 1) \cup \{x_1, x_2, \ldots, x_n\}) \times \mathbb{R} \neq \mathbb{R}^2$$

$(\mathbb{R}^2, \tau_1, \tau_2)$ is not $(1, 2) - a - c$.

However, we can easily prove that property (ii) (and also the equivalent property (iii)) of Theorem 3.2 does hold in $(\mathbb{R}^2, \tau_1, \tau_2)$. In fact if $\mathscr{B}$ is a τ_2-open filterbase on $\mathbb{R}^2$ and $(x_0, y_0) \in \tau_1-\text{adh}\, \mathscr{B}$ it is easy to see that for each $y \in \mathbb{R}$, (x_0, y) is also a τ_1-adherent point of $\mathscr{B}$. That means that each τ_2-open filterbase on $\mathbb{R}^2$ with non empty τ_1-adherence has an infinite number of τ_1-adherent points and hence $(\mathbb{R}^2, \tau_1, \tau_2)$ has property (ii) (and also property (iii)).

4. Minimal p-Hausdorff Spaces

The concept of minimal p-Hausdorff bitopological spaces was initiated by Raghavan and Reilly[8] as follows.

Definition 4.1—A bitopogical space (X, τ_1, τ_2) is called minimal p-Hausdorff if it is p-Hausdorff and if (X, τ_3, τ_4) is p-Hausdorff with $\tau_3 \subset \tau_1$ and $\tau_4 \subset \tau_2$, then $\tau_1 = \tau_3$ and $\tau_2 = \tau_4$.

Raghavan and Reilly[8] gave the following characterization of minimal p-Hausdorff spaces.

Theorem 4.2[8]—If (X, τ_1, τ_2) is p-Hausdorff, then the following are equivalent.

(a) (X, τ_1, τ_2) is minimal p-Hausdorff.

(b) For each τ_1-open filterbase $\mathscr{B}_1$ and for each τ_2-open filterbase $\mathscr{B}_2$ on X with $\tau_2\text{-adh}\, \mathscr{B}_1 = \tau_1\text{-adh}\, \mathscr{B}_2 = \{p\}$, $\mathscr{B}_1$ is τ_1-convergent to p and $\mathscr{B}_2$ is τ_2-convergent to p.

By the following result we show that there is no need of the sharing of the point p in the above Theorem.

Theorem 4.3—If (X, τ_1, τ_2) is p-Hausdorff, then the following are equivalent.

(i) (X, τ_1, τ_2) is minimal p-Hausdorff

(ii) Each τ_i-open filterbase $\mathscr{B}_i$ on X with a unique τ_j-adherent point x_i is τ_i-convergent to x_i $(i, j = 1, 2, i \neq j)$.

PROOF : We need only to prove that (b) of Theorem 4.2 implies (ii) of Theorem 4.3. Let $\mathcal{B}_i$ be a τ_i-open filterbase on X with τ_j-adh $\mathcal{B}_i = \{x_i\}$. Since (X, τ_1, τ_2) is p-Hausdorff, τ_i-adh $\tau_j(x_i) = \cap \{[V]_i : V \in \tau_j(x_i)\} = \{x_i\}$ (Reilly[9]). Therefore, by (b) of Theorem 4.2, $x_i \in \tau_i$-lim $\mathcal{B}_i$.

Our next result is a generalization of the following well-known Theorem 1.4 of Katětov[5] : A Hausdorff topological space is minimal Hausdorff iff it is almost compact and semiregular.

Theorem 4.4—A p-Hausdorff space (X, τ_1, τ_2) is minimal p-Hausdorff iff it is $p-a-c$ and p-semiregular.

PROOF : Let (X, τ_1, τ_2) be a minimal p-Hausdorff bitopological space and $\mathcal{B}_i$ a τ_i-open filterbase on X with τ_j-adh $\mathcal{B}_i = \{x_i\}$ $(i, j = 1, 2, i \neq j)$. By Theorem 4.3 and Proposition 1.12, $\mathcal{B}_i$ is θ_{ij}-convergent to x_i. Thus, by Theorem 3.2, (X, τ_1, τ_2) is $p-a-c$.

By Theorem 1.8, the p-semiregularization (X, τ_1^s, τ_2^s) of (X, τ_1, τ_2) is p-Hausdorff. Since now (X, τ_1, τ_2) is minimal p-Hausdorff, $\tau_1^s = \tau_1$ and $\tau_2^s = \tau_2$. Therefore, by Theorem 1.4, (X, τ_1, τ_2) is p-semiregular.

Conversely, let (X, τ_1, τ_2) be a p-Hausdorff $p-a-c$ and p-semiregular bitopological space and $\mathcal{B}_i$ a τ_i-open filterbase on X with τ_j-adh $\mathcal{B}_i = \{x_i\}$ $(i, j = 1, 2, i \neq j)$. By Theorem 3.2 and Lemma 1.13, $x_i \in \tau_i$-lim $\mathcal{B}_i$. Thus, by Theorem 4.3, (X, τ_1, τ_2) is minimal p-Hausdorff.

The following result is an immediate consequence of Theorems 1.7, 1.8, Proposition 2.2 and Theorem 4.4.

Theorem 4.5—If (X, τ_1, τ_2) is p-Hausdorff and $p-a-c$, then the p-semiregularization (X, τ_1^s, τ_2^s) of (X, τ_1, τ_2) is minimal p-Hausdorff.

Another well-known characterization of minimal Hausdorff topologisal spaces is the following : A Hausdorff topological spaceis minimal Hausdorff iff it is minimal semiregular Hausdorff. We note that necessity is obvious by Katětov's Theorem and that sufficiency is proved by Banaschewski[1] (p. 147).

Theorem 4.7 below is a bitopological analogue of the above characterization.

Definition 4.6—A bitopological space (X, τ_1, τ_2) is called minimal p-semiregular p-Hausdorff if it is p-semiregular and p-Hausdorff and if (X, τ_3, τ_4) is p semiregular and p-Hausdorff with $\tau_3 \subset \tau_1$ and $\tau_4 \subset \tau_2$, then $\tau_1 = \tau_3$ and $\tau_2 = \tau_4$.

Theorem 4.7—A bitopological space (X, τ_1, τ_2) is minimal p-Hausdorff iff it is minimal p-semiregular p-Hausdorff.

PROOF : Necessity follows from Theorem 4.4. To show sufficiency assume that (X, τ_1, τ_2) is a minimal p-semiregular p-Hausdorff space which is not minimal p-Hausdorff. Then there exist two topologies $\tau_3 \subset \tau_1$ and $\tau_4 \subset \tau_2$ with $\tau_3 \neq \tau_1$ or $\tau_4 \neq \tau_2$ such that (X, τ_3, τ_4) is p-Hausdorff. By Theorems 1.7 and 1.8, the p-semiregularization (X, τ_3^s, τ_4^s) of (X, τ_3, τ_4) is p-semiregular and p-Hausdorff. Since now $\tau_3^s \subset \tau_1$, $\tau_4^s \subset \tau_2$ and $\tau_3^s \neq \tau_1$ or $\tau_4^s \neq \tau_2$, (X, τ_1, τ_2) is not minimal p-semiregular p-Hausdorff and the contradiction completes the proof.

The following result follows immediately from the definitions and from the fact thae p-Hausdorffness is a projective and productive property[11].

Theorem 4.8—If the non empty bitopological product (X, τ_1, τ_2) of the family $((X_k, \tau_{1k}, \tau_{2k}))_{k \in K}$ [Swart[11]] is minimal p-Hausdorff, then each coordinate space $(X_k, \tau_{1k}, \tau_{2k})$ is minimal p-Hausdorff.

Finally, by the following example it is shown that the converse of Theorem 4.8 does not hold. So, the concept of minimal p-Hausdorff spaces is not product invariant.

Example 4.9—Let τ_1' be the discrete topology on $\mathbb{R}$, τ_2' the topology of finite complements on $\mathbb{R}$, $\tau_1'' = \tau_2'$ and $\tau_2'' = \tau_1'$. It is known[8] that the bitopological spaces $(\mathbb{R}, \tau_1', \tau_2')$ and $(\mathbb{R}, \tau_1'', \tau_2'')$ are minimal p-Hausdorff. Let now $(\mathbb{R}^2, \tau_1, \tau_2)$ be the bitopological product of the above spaces. The classes $\mathscr{B}_1 = \{\{x\} \times A : x \in \mathbb{R} \text{ and } \mathbb{R} - A \text{ finite}\}$ and $\mathscr{B}_2 = \{A \times \{x\} : x \in \mathbb{R} \text{ and } \mathbb{R} - A \text{ finite}\}$ are clearly bases for the topologies τ_1 and τ_2 respectively. If $p = (p_1, p_2)$ is a point of $\mathbb{R}^2$ it is easily proved that the classes $\mathscr{B}_1^* = \{A : A \in \mathscr{B}_1 \text{ and } p \notin A\} \cup \{(\mathbb{R} \times B) \cup C : \mathbb{R} - B$ finite, $C \in \mathscr{B}_1$ and $p \in C\}$, $\mathscr{B}_2^* = \{A : A \in \mathscr{B}_2 \text{ and } p \notin A\} \cup \{(B \times \mathbb{R}) \cup C : \mathbb{R} - B$ finite, $C \in \mathscr{B}_2$ and $p \in C\}$ are bases for the topologies τ_1^* and τ_2^* on $\mathbb{R}^2$, which are strictly weaker then τ_1 and τ_2 respectively. Since now $(\mathbb{R}^2, \tau_1^*, \tau_2^*)$ is p-Hausdorff it is obvious that the p-Hausdorff space $(\mathbb{R}^2, \tau_1, \tau_2)$ is not minimal p-Hausdorff.

The same conclusion can be obtained immediately by Theorem 4.4. In fact it is easily proved that each $B \in \mathscr{B}_1$ is (1,2)-regularly open and each $B \in \mathscr{B}_2$ (2,1)-regularly open and hence $(\mathbb{R}^2, \tau_1, \tau_2)$ is p-semiregular. However, since there exist a point $c = (0, 0) \in \mathbb{R}^2$, a τ_1-open cover $\mathscr{H} = \{\{x\} \times \mathbb{R} : x \in \mathbb{R}\}$ of $\mathbb{R}^2 - \{c\}$ and a τ_2-open neighbourhood $V = \mathbb{R} \times \{0\}$ of c, such that for each finite subset $\{x_1, x_2, \ldots x_n\}$ of $\mathbb{R}$, $[V]_{\tau_1} \cup (\cup \{[\{x_k\} \times \mathbb{R}]_{\tau_2} : k = 1, 2, \ldots, n\}) = (\mathbb{R} \times \{0\}) \cup (x_1, x_2, \ldots, x_n\} \times \mathbb{R}) \neq \mathbb{R}^2$,

$(\mathbb{R}^2, \tau_1, \tau_2)$ is not $p-a-c$. Therefore, by Theorem 4.4. $(\mathbb{R}^2, \tau_1, \tau_2)$ is not minimal p-Hausdorff.

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CERTAIN CLASSES OF p-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS II

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Let $P_p^*(\alpha, \beta)$ (p a fixed integer greater than zero, $0 \leqslant \alpha < p$ and $0 < \beta \leqslant 1$) denote the class of functions $f(z) = z^p - \sum_{n=1}^{\infty} | a_{p+n} | z^{p+n}$ analytic and p-valent in $|z| < 1$ for which

$$\left| \frac{\frac{f'(z)}{z^{p-1}} - p}{\frac{f'(z)}{z^{p-1}} + p - 2\alpha} \right| < \beta \; (|z| < 1).$$

Sharp results concerning coefficients, a distortion theorem and the radius of convexity for the class $P_p^*(\alpha, \beta)$ are determined. Furthermore, it is shown that the class $P_p^*(\alpha, \beta)$ is closed under convex linear combinations. The extreme points of the class $P_p^*(\alpha, \beta)$ are determined.

1. INTRODUCTION

Let S_p (p a fixed integer greater than zero) denote the class of functions of the form

$z^p + \sum_{n=1}^{\infty} | a_{p+n} | z^{p+n}$ that are analytic and p-valent in the unit disc $|z| < 1$. Also let T_p (p a fixed integer greater than zero) denote the subclass of S_p consisting of functions that are analytic and p-valent and can be expressed in the form $f(z) = z^p - \sum_{n=1}^{\infty} | a_{p+n} | z^{p+n}$.

A function $f \in T_p$ is in $T_p^*(\alpha, \beta)$ if and only if

$$\left| \frac{\frac{f'(z)}{z^{p-1}} - p}{\frac{f'(z)}{z^{p-1}} + p - 2\alpha} \right| < \beta \; (|z| < 1) \qquad \ldots(1.1)$$

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for α $(0 \leqslant \alpha < p)$ and $(0 < \beta \leqslant 1)$. We note that Gupta and Jain[3] studied the class $P_1^* (\alpha, \beta) = P^* (\alpha, \beta)$. Two sub-classes $S_p^* (\alpha, \beta)$ and $C_p^* (\alpha, \beta)$ of T_p, obtained by replacing $\frac{f'(z)}{z^{p-1}}$ with $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''(z)}{f'(z)}$ respectively in (1.1), have been studied by the author[1].

Schild[4] considered a subclass of T_1 consisting of those polynomials that have $|z| = 1$ as radius of univalence. For this class, he obtained a necessary and sufficient condition in terms of the coefficients, and with the aid of this he derived better results for certain quantities connected with the conformal mapping of univalent functions. Silverman[5] determined coefficient inequalities, as well as distortion, and covering theorems for the subclasses $S^* (\alpha)$ and $C^* (\alpha)$ of T_1, $0 \leqslant \alpha < 1$, classes of starlike functions of order α and convex functions of order α respectively. Gupta and Jain[2] determined sharp results concerning coefficients, the distortion of functions belonging to $S^* (\alpha, \beta)$ and $C^* (\alpha, \beta)$, $0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$, classes of starlike functions of order α and type β and convex functions of order α and type β respectively along with a representation formula for the functions in $S^* (\alpha, \beta)$. Furthermore, they have shown that the classes $S^* (\alpha, \beta)$ and $C^* (\alpha, \beta)$ are closed under convex linear combinations.

In this paper, sharp results concerning coefficients, a distortion theorem, and the radius of convexity for the class $P_p^* (\alpha, \beta)$ are determined. In the last section we assert that the class $P_p^* (\alpha, \beta)$ is closed under "arithmetic mean" and "convex linear combinations"; the extreme points of the class $P_p^* (\alpha, \beta)$ are also determined.

2. Coefficients Theorem

Theorem 1—A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}$ is in $P_p^* (\alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} (p + n)(1 + \beta) |a_{p+n}| \leqslant 2\beta (p - \alpha).$$

This result is sharp.

Proof : Let $|z| = 1$. Then

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| - \beta \left| \frac{f'(z)}{z^{p-1}} + p - 2\alpha \right| = \left| - \sum_{n=1}^{\infty} (p + n) |a_{p+n}| z^n \right|$$

$$- \beta \left| 2(p - \alpha) - \sum_{n=1}^{\infty} (p + n) |a_{p+n}| z^n \right|$$

$$\leqslant \sum_{n=1}^{\infty} (p + n)(1 + \beta) |a_{p+n}| - 2\beta (p - \alpha)$$

$$\leqslant 0, \text{ by hypothesis.}$$

Hence, by the maximum modulus theorem, $f \in P_p^*(\alpha, \beta)$.

For the converse, assume that

$$\left| \frac{\frac{f'(z)}{z^{p-1}} - p}{\frac{f'(z)}{z^{p-1}} + p - 2\alpha} \right| = \left| \left[- \sum_{n=1}^{\infty} (p+n) \mid a_{p+n} \mid z^n \right] \Big/ \left[2(p-\alpha) - \sum_{n=1}^{\infty} (p+n) \mid a_{p+n} \mid z^n \right] \right|$$

$$< \beta \; (\mid z < 1).$$

Since $\mid \mathrm{Re}(z) \mid \leqslant \mid z \mid$ for all z, we have

$$\mathrm{Re} \{[\sum_{n=1}^{\infty} (p+n) \mid a_{p+n} \mid z^n]/[2(p-\alpha) - \sum_{n=1}^{\infty} (p+n) \mid a_{p+n} \mid z^n] \}$$

$$< \beta. \qquad ...(2.1)$$

Choose values of z on the real axis so that $\frac{f'(z)}{z^{p-1}}$ is real. Upon clearing the denominator in (2.1) and letting $z \to 1$ through real values, we obtain

$$\sum_{n=1}^{\infty} (p+n) \mid a_{p+n} \mid \leqslant 2\beta(p-\alpha) - \beta \sum_{n=1}^{\infty} (p+n) \mid a_{p+n} \mid .$$

This gives the required condition.

The result is sharp, the extremal function being

$$f(z) = z^p - \frac{2\beta(p-\alpha)}{(p+n)(1+\beta)} z^{p+n}.$$

3. Distortion Theorem

Theorem 2— If $f \in P_p^*(\alpha, \beta)$, then

$$r^p - \frac{2\beta(p-\alpha)}{(p+1)(1+\beta)} r^{p+1} \leqslant \mid f(z) \mid \leqslant r^p + \frac{2\beta(p-\alpha)}{(p+1)(1+\beta)} r^{p+1}$$

$$(\mid z \mid = r) \qquad ...(3.1)$$

and

$$pr^{p-1} - \frac{2\beta(p-\alpha)}{(1+\beta)} r^p \leqslant \mid f'(z) \mid \leqslant pr^{p-1} + \frac{2\beta(p-\alpha)}{(1+\beta)} r^p$$

$$(\mid z \mid = r). \qquad ...(3.2)$$

PROOF : In view of Theorem 1, we have

$$\sum_{n=1}^{\infty} | a_{p+n} | \leqslant \frac{2\beta (p-\alpha)}{(p+1)(1+\beta)} .$$

Hence

$$| f(z) | \leqslant r^p + \sum_{n=1}^{\infty} | a_{p+n} | r^{p+1}$$

$$\leqslant r^p + \frac{2\beta (p-\alpha)}{(p+1)(1+\beta)} r^{p+1}$$

and

$$| f(z) \geqslant r^p - \sum_{n=1}^{\infty} | a_{p+n} | r^{p+1}$$

$$\geqslant r^p - \frac{2\beta (p-\alpha)}{(p+1)(1+\beta)} r^{p+1}.$$

Thus (3.1) follows.

Also,

$$| f'(z) | \leqslant pr^{p-1} + \sum_{n=1}^{\infty} (p+n) | a_{p+n} | r^p$$

$$\leqslant pr^{p-1} + \frac{2\beta (p-\alpha)}{(1+\beta)} r^p$$

and

$$| f'(z) | \geqslant pr^{p-1} - \sum_{n=1}^{\infty} (p+n) | a_{p+n} | r^p$$

$$\geqslant pr^{p-1} - \frac{2\beta (p-\alpha)}{(1+\beta)} r^p.$$

This completes the proof of the theorem.

Remark : The bounds in (3.1) and (3.2) are sharp since the equalities are attained for the function

$$f(z) = z^p - \frac{2\beta (p-\alpha)}{(p+1)(1+\beta)} z^{p+1} \; (z = \pm r).$$

Theorem 3—Let $f \in P_p^*(\alpha, \beta)$. Then the disc $| z | < 1$ is mapped onto a domain that contains the disc

$|w| < \dfrac{(p+1)+\beta(1-p+2\alpha)}{(p+1)(1+\beta)}$. The result is sharp with extremal function $z^p - \dfrac{2\beta(p-\alpha)}{(p+1)(1+\beta)} z^{p+1}$.

PROOF : The result follow upon letting $r \to 1$ in (3.1).

Theorem 4:—If $f \in P_p^*(\alpha, \beta)$, then f is convex in the disc $|z| < r = r(\alpha, \beta, p)$, where

$$r(\alpha, \beta, p) = \inf_n \left\{ \frac{p^2(1+\beta)}{2\beta(p+n)(p-\alpha)} \right\}^{1/n}, \quad n = 1, 2, 3, \ldots.$$

This result is sharp, the extermal function being of the form

$$f(z) = z^p - \frac{2\beta(p-\alpha)}{(p+n)(1+\beta)} z^{p+n}.$$

PROOF : It suffices to show that $\left|(1 + \dfrac{z f''(z)}{f'(z)}) - p\right| \leqslant p$ for $|z| \leqslant 1$. First, we note that

$$\left|\left(1 + \frac{z f''(z)}{f'(z)}\right) - p\right| = \left|\frac{z f''(z) + (1-p) f'(z)}{f'(z)}\right|$$

$$\leqslant \{\sum_{n=1}^{\infty} n(p+n)\,|a_{p+n}|\,|z|^n\}/\{p - \sum_{n=1}^{\infty}(p+n)\,|a_{n+n}|\,|z|^n\}.$$

Thus the result follows if

$$\sum_{n=1}^{\infty} n(p+n)\,|a_{p+n}||z|^n \leqslant p\{p - \sum_{n=1}^{\infty}(p+n)\,|a_{p+n}||z|^n\}$$

which is equivalent to

$$\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)^2 |a_{p+n}||z|^n \leqslant 1.$$

But by Theorem 1,

$$\sum_{n=1}^{\infty}(p+n)(1+\beta)\,|a_{p+n}| \leqslant 2\beta(p-\alpha).$$

Hence f is convex if

$$\left(\frac{p+n}{p}\right)^2 |z|^n \leqslant \frac{(p+n)(1+\beta)}{2\beta(p-\alpha)}, \quad n = 1, 2, 3, \ldots.$$

that is,

$$| z | \leqslant \left\{ \frac{p^2 (1 + \beta)}{2\beta (p + n) (p - \alpha)} \right\}^{1/n}, \; n = 1, 2, 3, \ldots.$$

This completes the proof.

4. Closure Theorems

In this section we assert that the class $P_p^* (\alpha, \beta)$ is closed under "arithmetic mean" and "convex linear combinations".

Theorem 5— If $f(z) = z^p - \sum_{n=1}^{\infty} | a_{p+n} | z^{p+n}$ and

$$g(z) = z^p - \sum_{n=1}^{\infty} | b_{p+n} | z^{p+n} \text{ are in } P_p^* (\alpha, \beta), \text{ then}$$

$$h(z) = z^p - \tfrac{1}{2} \sum_{n=1}^{\infty} | a_{p+n} + b_{p+n} | z^{p+n}$$

is also in $P_p^* (\alpha, \beta)$.

Theorem 6—Let $f_p(z) = z^p$, $f_{p+n}(z) = z^p - \dfrac{2\beta (p - \alpha)}{(p + n)(1 + \beta)} z^{p+n}$, $(n = 1, 2, \ldots)$. Then $f \in P_p^* (\alpha, \beta)$ if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z)$, where $\lambda_{p+n} \geqslant 0$, and $\sum_{n=0}^{\infty} \lambda_{p+n} = 1$.

Proofs of Theorems 5 and 6 follow along the same lines as the proofs of Theorems 8 and 9 in Aouf[1]. The details are omitted.

Corollary —The extreme points of $P_p^* (\alpha, \beta)$ are the functions $f_p(z) = z^p$ and $f_{p+n}(z) = z^p - \dfrac{2\beta (p - \alpha)}{(p+n)(1 + \beta)} z^{p+n}$, $(n = 1, 2, 3, \ldots)$.

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SOME QUADRATIC TRANSFORMATIONS OF BASIC HYPERGEOMETRIC SERIES AND IDENTITIES IN RAMANUJAN'S 'LOST' NOTE BOOK

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One of the identities of Ramanujan's 'lost' note book on Partial theta function and his identities related to Stacks with summits were proved by Andrews by using q-difference equations as they did not fit in the known transformation theory of basic hypergeometric series. In this paper quadratic transformations of basic hypergeometrie series are obtained which besides containing Ramanujan's identities as special case also yield more general analytical identities of the same nature.

1. INTRODUCTION

Andrews[6] had proved identities on partial theta functions, mentioned without proof, in Ramanujan's 'Lost' note book (see Andrews[5] for historical background of the document). The elegance of Andrews proof lies in his observation that practically all the Ramanujan's identities on partial theta functions, except the identity [Andrews[6], (1.2) $_R$]

$$\sum_{n=0}^{\infty} \frac{[q^{1+n}; q]_n\, q^n}{[-aq, -q/a; q]_n} = (1+a) \sum_{n=0}^{\infty} (-a)^n q^{n^2+n}$$

$$-a \sum_{n=0}^{\infty} \frac{(-a^3)^n (1+aq^{2n+1})\, q^{3n^2+2n}}{[-aq, -q/a; q]_\infty} \qquad \ldots(1.1)$$

are all special cases of a single transformation of basic hypergeometric series. Agarwal[1] had noted that the transformation of basic hypergeometric series, proved in an ingenious way by Andrews[6] for obtaining Ramanujan's identities on partial theta functions, is a special case of a three-term relation between $_3\phi_2$'s given by Sears[14]. This observa-

tion lead Agarwal[1] to obtain not only three term relation but also four-term relations between partial theta functions.

For the innocent looking identity (1.1) Andrews[6] was unable to find any generalization and this forced him to prove (1.1) by a lengthy tour deforce. According to his own reckoning it would be of interest to find a generalization of (1.1) which would place it in an appropriate place in the transformation theory of basic hypergeometric series. In section 3 we show that (1.1) could be deduced from the quadratic transformation

$$ {}_3\phi_2\left[\begin{matrix} a, c, d; q; \dfrac{aqx}{cd} \\ aq/c, \quad aq/d \end{matrix}\right] = \frac{[ax; q]_\infty}{[x; q]_\infty} $$

$$ {}_5\phi_4\left[\begin{matrix} \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, aq/cd; q; q \\ aq/c, aq/d, ax, q/x \end{matrix}\right] $$

$$ + \frac{[a, aq/cd, axq/c; axq/d; q]_\infty}{[aq/c, aq/d, aqx/cd, 1/x; q]_\infty} $$

$$ {}_5\phi_4\left[\begin{matrix} x\sqrt{a}, -x\sqrt{a}, x\sqrt{aq}, -x\sqrt{aq}, axq/cd; q; q \\ axq/c, axq/d, xq, ax^2 \end{matrix}\right]. \qquad \ldots(1.2) $$

Throughout the paper we will assume $|q| < 1$, $[a; q]_n = (1-a)(1-aq) \ldots (1-aq^{n-1})$, $[a;q]_0 = 1$, $[a; q]_\infty = \prod_{j=0}^{\infty} (1-aq^j)$, abbreviate $[a_1; q]_n [a_2; q]_n \ldots [a_r; q]_n$ by $[a_1, a_2, \ldots, a_r; q]_n$ and define the basic hypergeometric series as

$$ {}_{p+1}\phi_{p+r}\left[\begin{matrix} a_1, a_2, \ldots, a_{p+1}; q; x \\ b_1, b_2, \ldots, b_{p+r} \end{matrix}\right] $$

$$ = \sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_{p+1}; q]_n (-)^{nr} x^r q^{nr(n-1)/2}}{[q, b_1, b_2, \ldots, b_{p+r}; q]_n}. $$

The basic hypergeometric series ${}_{p+1}\phi_{p+r}$ converges for all positive integral values of r and for all x, except when $r = 0$ it converges only for $|x| < 1$. Following Askey-Wilson[8] the basic hypergeometric series ${}_{p+1}\phi_p\left[\begin{matrix} a_1, a_2, \ldots, a_{p+1}; q; x \\ b_1, b_2, \ldots, b_p \end{matrix}\right]$ would be called balanced if $qa_1 a_2, \ldots, a_{p+1} = b_1 b_2 \ldots b_p$, well-poised if $qa_1 = a_2 b_1 = a_3 b_2 = \ldots = a_{p+1} b_p$ and nearly-poised if any one of the above equalities breaks down. Furthermore, the well-poised basic hypergeometric series

$$ {}_{p+3}\phi_{p+2}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a_1, a_2, \ldots, a_p; q; x \\ \sqrt{a}, -\sqrt{a}, \dfrac{qa}{a_1}, \dfrac{qa}{a_2}, \ldots, \dfrac{qa}{a_p} \end{matrix}\right] $$

would be abbreviated by ${}_{p+3}W_{p+2}\,[a; a_1\, a_2, \ldots, a_p; q; x]$ and called as a very well-poised basic hypergeometric series.

It is also shown in Section 3 that the quadratic transformation

$$
{}_4\phi_3\begin{bmatrix} a, q\sqrt{a}, b, c; q; xq/bc \\ \sqrt{a}, aq/b, aqc \end{bmatrix} = \frac{[xq, -x/\sqrt{a}; q]_\infty}{[x/a, -xq/\sqrt{a}; q]_\infty} .
$$

$$
\times\, {}_5\phi_4\begin{bmatrix} \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -\sqrt{a}, aq/bc; q; q \\ aq/b, aq/c, xq, aq/x \end{bmatrix}
$$

$$
+ \frac{[aq, xq/b, xq/c, aq/bc, xq/a, xq/\sqrt{a}, -\sqrt{a}; q]_\infty}{[aq/b, aq/c, xq/bc, aq/x, x/a, -xq/\sqrt{a}, -q\sqrt{a}; q]_\infty}
$$

$$
\times\, {}_5\phi_4\begin{bmatrix} x\sqrt{q}/\sqrt{a}, -x\,q/\sqrt{a}, xq/\sqrt{a}, -x/\sqrt{a}, xq/bc; q; q \\ xq/a, x^2q/a, xq/b, xq/c \end{bmatrix} \qquad \ldots(1.3)
$$

could be used to obtain the identity (c. f. (1.1))

$$
\sum_{n=0}^{\infty} \frac{[q; q^2]_n\,[-q; q]_{n-1}\, q^n}{[aq, q/a; q]_n} = \frac{1-a}{1+a} \sum_{n=0}^{\infty} a^n\, q^{n^2}
$$

$$
- \sum_{n=0}^{\infty} \frac{(1 - a^2\, q^{4n+2})\, a^{3n}\, q^{3n^2+n}}{[-a, q/a; q]_\infty} . \qquad \ldots(1.4)
$$

Furthermore, the transformation between very well-poised ${}_5W_4$ and balanced ${}_5\phi_4$ viz.,

$$
{}_5W_4\,[a; b, c; q; x\sqrt{aq}\,/bc] = (1 - x^2)\, \frac{[xq\sqrt{aq}; q]_\infty}{[x/\sqrt{aq}; q]_\infty}
$$

$$
\times\, {}_5\phi_4\begin{bmatrix} \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a}, aq/bc; q; q \\ aq/b, aq/c, xq\sqrt{aq}, q\sqrt{aq}/x \end{bmatrix}
$$

$$
+ \frac{[aq, aq/bc, x\sqrt{aq}/b, x\sqrt{aq}/c; q]_\infty}{[aq/b, aq/c, x\sqrt{aq}/bc, \sqrt{aq}/x; q]_\infty}
$$

$$
\times\, {}_5\phi_4\begin{bmatrix} x, -x, x\sqrt{q}, -x\sqrt{q}, x\sqrt{aq}/bc; q; q \\ x^2q, x\sqrt{aq}/b, x\sqrt{aq}/c, x\sqrt{q}/\sqrt{a} \end{bmatrix} \qquad \ldots(1.5)
$$

yield the following identity analogue to (1.1) and (1.4)

$$
(1+a) \sum_{n=0}^{\infty} \frac{[q; q]_{2n+1}\, q^{n+1}}{[q; q]_n\, [aq, q/a; q]_{n+1}} = \sum_{n=0}^{\infty} \frac{(1 - a^2\, q^{4n+2})\, a^{3n+1}\, q^{3n^2+n}}{[aq, 1/a; q]_\infty}
$$

$$
- \sum_{n=0}^{\infty} (1 - q^{2n+1})\, a^{n+1}\, q^{n^2}. \qquad \ldots(1.6)
$$

In fact (1.2) and (1.5) are known quadatic transformations of Gasper and Rahman[12] [(1.27) and (1.35) respectively]. They deduced (1.2) and (1.5) by letting $n \to \infty$ in Bailey's transformations[10] [(1) and (2)] between terminating nearly-poised basic hypergeometric series and terminating very well-poised basic hypergeometric series. The transformation (1.2) is a non-terminating version of Sears-Carlitz formula [Sears[15] 4.1, Carlitz[11], (2.4)]. As the proofs of (1.2), (1.3) and (1.5) follow on the same lines, hence, for completeness sake, alternative proofs of (1.2) and (1.5) are given in section 2, along with that of (1.3).

Andrews[7] had proved the lemmas ;

Lemma 1A—For each nonnegative integer m.

$$[q;\, q^2]_m \sum_{n=0}^{2m} \frac{(-)^n}{[q;\, q]_n [a;\, q]_{2m-n}} = \sum_{n=0}^{m} \frac{(-)^n q^{n^2}}{[q^2;\, q^2]_{m-n}\, [aq;\, q^2]_n} \qquad \ldots(1.7)$$

Lemma 2A—For each positive integer m,

$$[q;\, q^2]_m \sum_{n=0}^{2m-1} \frac{(-)^n}{[q;\, q]_n\, [a;\, q]_{2m-n-1}} = \left(1 - \frac{a}{q}\right) \sum_{n=1}^{m} \frac{(-)^n q^{n^2}}{[q^2;\, q^2]_{m-n}\, [a;\, q^2]_n} \qquad \ldots(1.8)$$

for proving some identities of 'lost' note book of Ramanujan. Andrews[7] (p. 34) had remarked that Professor R. Askey had conjectured that the above Lemmas 1A and 2A should follow from some quadratic transformation of basic hypergeometric series. In Section 4, Professor Askey's conjecture is proved by obtaining the quadratic transformation

$${}_5\phi_4 \left[\begin{matrix} a^2,\, b^2,\, q,\, x,\, -x;\, q;\, q \\ ab\sqrt{q},\, -ab\sqrt{q},\, c,\, x^2 q/c \end{matrix} \right] = {}_4\phi_3 \left[\begin{matrix} a^2,\, b^2,\, q^2,\, x^2;\, q^2;\, q^2 \\ a^2b^2q,\, cq,\, x^2q^2/c \end{matrix} \right] \qquad \ldots(1.9)$$

(a^2, b^2, or x is of the form q^{-n}, n a non-negative integer). The transformation (1.9) for $a = q^{-m}$, $b \to \infty$, $x \to 0$ reduces to Lemma 1A, whereas for $a = q^{1/2-m}$, $b \to \infty$, $x \to 0$, it gives Lemma 2A and for $b \to \infty$, $a = q^{-m}$, c, $x \to 0$, (1.9) yields yet another formula of Andrews[6] (4.3) which was his key result for proving an identity of Ramanujan on partial theta functions. Using (1.9), following two generalizations of Ramanujan's identities [Andrews[7], (1.10) (1.11)] are obtained

$$\sum_{n=0}^{\infty} \frac{[x^2;\, q^2]_n\, [q^{1+n};\, q]_\infty\, q^{n(n+1)/2}}{[-x^2,\, -q;\, q]_n} +$$

(equation continued on p. 772)

$$+\sum_{n=0}^{\infty}\frac{[x^2;q^2]_n\,[-q^{1+n};q]_\infty\,(-)^n q^{n(n+1)/2}}{[-x^2,-q;q]_n}$$

$$=2\,[-q;q]_\infty\sum_{n=0}^{\infty}\frac{[x^2;q^2]_n\,(-q)^{n(n+1)/2}}{[-q^2;q^2]_n\,[x^2;-q]_n} \qquad \ldots(1.10)$$

$$\sum_{n=0}^{\infty}\frac{[x^2;q^2]_{n+1}\,[q^{1+n};q]_\infty\,q^{n(n+1)/2}}{[-q;q]_n\,[-x^2q;q]_{n+1}}$$

$$-\sum_{n=0}^{\infty}\frac{[x^2;q^2]_{n+1}\,[-q^{n+1};q]_\infty\,(-)^n q^{n/2(n+1)}}{[-q;q^2]_n\,[-x^2q;q]_{n+1}}$$

$$=4\,[-q;q]_\infty\sum_{n=1}^{\infty}\frac{[x^2;q^2]_{2n}\,(-)^n q^{2n^2}}{[-q;q^2]_{2n}\,[x^4q^4;q^4]_n} \qquad \ldots(1.11)$$

which for $x\to 0$ reduce to Ramanujan's identities Andrews[7], (1.10) and (1.11).

2. Proofs of (1.2), (1.3) and (1.5)

In the transformation connecting non-terminating nearly-poised ${}_5\phi_4$ series of the first kind and very well-poised ${}_{12}\phi_{11}$ [Verma and Jain[10], 7.1] setting $e=a^2q/dk$, $c=1$, we get

$$\begin{aligned}&{}_{10}W_9\,[k;k/a,\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},f,k^2q/af;q;q]\\&+\frac{[kq/af,kq,a,f,kq^2/f,k/a,k^4q^3/a^3f^2;q]_\infty}{[kq/f,aq,k^2q/a,af/kq,k^3q^3/a^2f^2,k^2q/a^2f,kq/a;q]_\infty}\\&\times{}_{10}W_9\Big[\frac{k^3q^2}{a^2f^2};\frac{k^2q}{a^2f},\frac{kq}{f\sqrt{a}},-\frac{kq}{f\sqrt{a}},\frac{kq^{3/2}}{f\sqrt{a}},\\&\qquad-\frac{kq^{3/2}}{f\sqrt{a}},\frac{kq}{a},\frac{k^2q}{af};q;q\Big]\\&=\frac{[kq/af,k^2q/a^2,kq,k^2q/af;q]_\infty}{[kq/f,kq/a,k^2q/a,k^2q/a^2f;q]_\infty}.\end{aligned} \qquad \ldots(2.1)$$

In (2.1) replacing k by afx/q and then letting $f\to 0$, we get

$${}_4\phi_3\left[\begin{matrix}\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q;q\\aq,ax,q/x\end{matrix}\right]+\frac{[x,a,aqx;q]_\infty}{[aq,ax,1/x;q]_\infty}$$

(equation continued on p. 773)

$$ {}_4\phi_3\left[\begin{matrix} x\sqrt{a},\ -x\sqrt{a},\ x\sqrt{aq},\ -x\sqrt{aq};\ q;\ q \\ axq,\ ax^2,\ xq \end{matrix}\right] = \frac{[x;q]_\infty}{[ax;q]_\infty}. \quad \ldots(2.2) $$

In (2.2) replacing a and x by aq^{2n} and xq^{-n} respectively and multiplying both sides by

$$ \frac{[aq^n, xaq^n, q^{1+n}/x; q]_\infty\ [a, d, c; q]_n\ q^{n(n+1)/2}}{[a;q]_\infty\ [q, aq/c, aq/d; q]_n} (-aq/cd)^n $$

and summing with respect to n from 0 to ∞, we get (1.2), on diagonalizing the double sums on the left-hand-side and summing the resulting inner ${}_6\phi_5$ by Sears[15] (6.2).

Proof of (1.3)—In the transformation [Verma and Jain[16], 7.7] between non terminating nearly poised ${}_6\phi_5$ of the first kind and very well-poised ${}_{12}\psi_{11}$: $(e = a^2 q/kcd)$

$$ \frac{[\sqrt{a}, aq/c, aq/d, aq/e, a^2 fq/k^2; q]_\infty}{[q\sqrt{a}, a, c, d, e, f; q]_\infty}\ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, c, d, e, f;\ q;\ q \\ \sqrt{a}, aq/c, aq/d, aq/e, a^2 q f/k^2 \end{matrix}\right] $$

$$ - \frac{k^2}{a^2 f} \frac{[k^2 q/afc, k^2q/afd, k^2 q/afe, k^2q/a^2f, k^2/fa^{3/2}; q]_\infty}{[k^2/af, k^2 c/a^2 f, k^2 d/a^2 f, k^2 e/a^2 f, k^2/a^2, k^2q/fa^{3/2}; q]_\infty} $$

$$ \times\ {}_6\phi_5\left[\begin{matrix} k^2/a^2, k^2q/fa^{3/2}, k^2c/a^2 f, k^2d/a^2f, k^2e/a^2f, k^2/af;\ q;\ q \\ k^2/fa^{3/2}, k^2q/afc, k^2q/afd, k^2q/afe, k^2q/a^2f \end{matrix}\right] $$

$$ = \frac{[a^2fq/k^2, k^2/a^2f, k/a, k^2/a, aq/c, aq/d, aq/e, kq/f, \sqrt{a}, -k/f\sqrt{a}, -kq/\sqrt{a}; q]_\infty}{[f, c, d, e, a, k/af, k^2/a^2, kq, k^2/af, q\sqrt{a}, -\dfrac{kq}{f\sqrt{a}}, -\dfrac{k}{\sqrt{a}}; q]_\infty} $$

$$ {}_{12}W_{11}\,[k; kc/a, kd/a, ke/a, f, k^2/af, \sqrt{aq}, q\sqrt{a}, -\sqrt{aq}, -\sqrt{a}; q; q] $$

$$ + \frac{[kc/a, kd/a, ke/a, a^2qf/k^2, k^2/a^2f, kq/cf, kq/df, kq/ef, k^4q/a^3f^2, k^2/fa^{3/2}; q]_\infty}{[c, d, e, af/k, k^2/a^2, k^2/af, k^2c/a^2f, k^2d/a^2f, k^2 e/a^2f, k^3q/a f^2, k^2q/fa^{3/2}; q]_\infty} $$

$$ {}_{12}W_{11}\,[k^3/a^2 f^2, k^2 c/a^2 f, k^2 d/a^2 f, k^2 e/a^2 f, k/a, k^2/af, k\sqrt{q}/f\sqrt{a}, $$

$$ -k\sqrt{q}/f\sqrt{a}, kq/f\sqrt{a}, -k/f\sqrt{a}; q; q], $$

multiplying both sides by $[c;q]_\infty$ ond setting $c = 1$ (such that $e = a^2 q/kd$), we get

$$ {}_{10}W_9\,[k; k/a, f, k^2/af, \sqrt{aq} - \sqrt{aq}, q\sqrt{a}, -\sqrt{a}; q; q] $$

$$ + \frac{[kq, a, f, q\sqrt{a}, k/af, kq/f\sqrt{a}, -k/\sqrt{a}. k^2/fa^{3/2}, k^2d/a^2f, k^4q/a^3f^2; q]_\infty}{[aq, \sqrt{a}, af/k, -k/f\sqrt{a}, -kq/\sqrt{a}, k^2/a, k^2/a^2f, k^2d/a^2f, k^2q/fa^{3/2}, \dfrac{k^3q}{a^2f^2}; q]_\infty}. $$

$$ {}_{10}W_9\left[\frac{k^3}{a^2f^2};\frac{k^2}{a^2f},\frac{k}{a},\frac{k^2}{af},\frac{k}{f}\sqrt{\frac{q}{a}},-\frac{k}{f}\sqrt{\frac{q}{a}},\frac{kq}{f\sqrt{a}},-\frac{k}{f\sqrt{a}};q;q\right] $$

$$ = \frac{[k/af, k^2/a^2kq, k^2/af, -kq/f\sqrt{a}; -k\sqrt{a}; q]_\infty}{[k^2/a^2f, k/a, k^2a, kq/f, -k/f\sqrt{a}, -kq/\sqrt{a}; q]_\infty}. \quad \ldots(2.3) $$

In (2.3) setting $k = xf$ and letting $f \to 0$, we get on some reduction

$$\frac{[aq, xq, aq/x; q]_\infty}{[\sqrt{aq}, q\sqrt{a}, -\sqrt{aq}, -\sqrt{a}; q]_\infty} {}_4\phi_3 \left[\begin{matrix}\sqrt{aq}, q\sqrt{a}, -\sqrt{aq}, -\sqrt{a}; q; q \\ aq, xq, aq/x\end{matrix}\right]$$

$$-\frac{x}{a}\frac{[xq, xq/a, xq/\sqrt{a}; q]_\infty}{[-x/\sqrt{a}; q]_\infty} {}_4\phi_3 \left[\begin{matrix}x\sqrt{q}/\sqrt{a}, -x\sqrt{q}/\sqrt{a}, xq/\sqrt{a}; -x\sqrt{a}; q; q \\ xq, xq/a, x^2q/a\end{matrix}\right]$$

$$= \frac{[aq/x, x/a, -xq/\sqrt{a}, -q\sqrt{a}; q]_\infty}{[-\sqrt{a}, -x/\sqrt{a}; q]_\infty}. \qquad \ldots(2.4)$$

In (2.4) replacing a and x by aq^{2r} and xq^r respectively and multiplying both sides by

$$\frac{[a, b, c; q]_r \; [aq^2; q^2]_r}{[q, aq/b, aq/c; q]_r \, [a; q^2]_r} \left(-\frac{a}{bc}\right)^r q^{r(r+3)/2}$$

summing with respect to r from 0 to ∞, we get (1.3) on diagonalizing both the double sums on the left-hand side and summing the resulting inner terminating series.

Proof of (1.5)—In the transformation [Verma and Jain[16], § 7, equation of page 25-26] between non-terminating nearly-poised ${}_7\phi_6$ of the first kind and very well-poised ${}_{12}W_{11}$ first replacing f by f^2 and then setting $c = 1$ and $e = \dfrac{a^2 q}{dk}$ we get

$${}_{10}W_9\,[k; k/a, k^2/faq, f, \sqrt{aq}, -\sqrt{aq}, q\sqrt{a}; -q\sqrt{a}; q; q]$$

$$+\frac{\left[f, kq, \frac{k}{a}, \frac{k}{f}, \frac{k^2d}{fa^2q}, \frac{k^4}{f^2a^3q^2}, \frac{k}{afq}, \frac{k^3}{af^2}; q\right]_\infty}{\left[\frac{kq}{f}, \frac{k^2}{a}, \frac{k}{aq}, \frac{k^2}{af^2q}, \frac{afq}{k}, \frac{k^2}{a^2fq}, \frac{k^2d}{a^2fq}, \frac{k^2}{a^2f^2q}; q\right]_\infty}$$

$${}_{10}W_9\left[\frac{k^3}{a^2f^2q^2}; \frac{k^2}{a^2fq}, \frac{k}{f\sqrt{aq}}, -\frac{k}{f\sqrt{aq}}, \frac{k}{f\sqrt{a}}, -\frac{k}{f\sqrt{a}}, \frac{k^2}{faq}, \frac{k}{aq}; q; q\right]$$

$$= \left[\frac{k^2}{a^2q}, \frac{k^2}{afq}, \frac{k^2}{afq^2}, \frac{k}{af^2}, kq; q\right]_\infty \Bigg/$$

$$\left[\frac{kq}{f}, \frac{k^2}{a}, \frac{k}{aq}, \frac{k^2}{af^2q}, \frac{k^2}{a^2fq}; q\right]_\infty. \qquad \ldots(2.5)$$

In (2.5) replacing k by $xf\sqrt{aq}$ and then betting $f \to 0$ we get

$$[xq\sqrt{aq}, q\sqrt{aq}/x; q]_\infty \, {}_4\phi_3 \left[\begin{matrix}\sqrt{aq}, -\sqrt{aq}, q\sqrt{a}, -q\sqrt{a}; q; q \\ aq, xq\sqrt{aq}, q\sqrt{aq}/x\end{matrix}\right]$$

$$-\frac{x}{\sqrt{aq}}\frac{[x\sqrt{aq}, x\sqrt{q}/\sqrt{a}, x^2q; q]_\infty}{[x^2; q]_\infty}$$

$$\times\, {}_4\phi_3 \left[\begin{matrix}x, -x, x\sqrt{q}, -x\sqrt{aq}; q; q \\ x\sqrt{aq}, x\sqrt{q}/\sqrt{a}, x^2q\end{matrix}\right]$$

(equation contiuned on p. 775)

$$= \frac{[x/\sqrt{aq}, x^2q, q\sqrt{aq}/x; q]_\infty}{[x^2; q]_\infty}. \qquad ...(2.6)$$

In (2.6) replacing a by aq^{2r}, and then multiplying both sides by

$$\frac{[a, b, c; q]_r [aq^2; q^2]_r}{[q, aq/b, aq/c; q]_r [a; q^2]_r} \left(-\frac{a}{bc}\right)^r q^{r(r+3)/2}$$

and summing with respect to r from 0 to ∞, we get (1.5) on diagonalizing both the double series on the left-hand side and summing the inter terminating basic hypergeometric series.

3. Proofs of (1.1), (1.4) and (1.6)

We begin this section by first proving the simple quadratic transformations :

$$[q; q^2]_\infty \; {}_3\phi_2 \left[\begin{matrix} a, b, -b; q; q \\ b, 0 \end{matrix}\right] = [aq; q^2]_\infty \; {}_2\phi_1 \left[\begin{matrix} a, 0; q^2; q^2 \\ b^2q \end{matrix}\right] \qquad ...(3.1)$$

$$[-aq, q; q]_\infty \; {}_3\phi_2 \left[\begin{matrix} a\sqrt{q}, -a\sqrt{q}. aq; q; q \\ a^2 q, 0 \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} (1 + aq^{2n+1}) (-a^3)^n q^{3n^2+2n} \qquad ...(3.2)$$

$$[aq, q; q]_\infty \; {}_3\phi_2 \left[\begin{matrix} a\sqrt{q}, -a\sqrt{q}, -a; q; q \\ a^2 q, 0 \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} (1 - a^2 q^{4n+2}) a^{3n} q^{3n^2+n}. \qquad ...(3.3)$$

Transformation (3.1) is very similar to one given by Jain[13] [4.1].

Proof of (3.1)—Using q-Vandermonde's theorem the left-hand side of (3.1) becomes [Sears[15], 4.2]

$$= \sum_{n=0}^{\infty} \frac{[a; q]_n}{[q; q]_n} q^{n(n+1)/2} \; {}_2\phi_1 \left[\begin{matrix} q^{-n}, q^{1-n}; q^2; q^2 \\ b^2q \end{matrix}\right] [q; q^2]_\infty$$

$$= \sum_{r=0}^{\infty} \frac{[a; q]_{2r} \, q^{2r}}{[q^2, b^2 q; q]_r} \; {}_1\phi_1 \left[\begin{matrix} aq^{2r}; q; -q \\ 0 \end{matrix}\right] [q; q^2]_\infty$$

summing the inner series by the limiting case of the summation formula [Sears[15] (6.2)]

$${}_6W_5 \left[a, b, c, d; q; \frac{aq}{bcd}\right] = \frac{[aq, aq/bc, aq/cd, aq/bd; q]_\infty}{[aq/b, aq/c, aq/d, aq/bcd; q]_\infty} \qquad ...(3.4)$$

(taking $b = \sqrt{a}$, $c = -\sqrt{a}$, $d \to \infty$), we we get (3.1).

Proof of (3.2)—In view of the transformation (3.1) (with a and b be replaced by aq and $a\sqrt{q}$ respectively) the left hand-side of (3.2) (= S, say) becomes:

$$= [-aq, q; q]_\infty \frac{[aq^2: q^2]_\infty}{[q: q^2]_\infty} {}_2\phi_1 \left[\begin{matrix} aq, 0; q^2; q^2 \\ a^2q^2 \end{matrix} \right]. \qquad ...(3.5)$$

Next using the transformation

$$ {}_1\phi_1 \left[\begin{matrix} aq; q^2; aq^3 \\ a^2 q^2 \end{matrix} \right] = [q^2; q^2]_\infty \, {}_2\phi_1 \left[\begin{matrix} aq, 0; q^2; q^2 \\ a^2 q^2 \end{matrix} \right]$$

obtained as a special case of q-Euler's transformation (Sears[14], p. 176)

$$ {}_2\phi_1 \left[\begin{matrix} a, b; q; \frac{ez}{ab} \\ e \end{matrix} \right] = \frac{[z; q]_\infty}{[ez/ab; q]_\infty} \, {}_2\phi_1 \left[\begin{matrix} e/a, e/b; q; z \\ e \end{matrix} \right] \qquad ...(3.6)$$

(by first q replacing by q^2 then replacing a, e, z by aq, $a^2 q^2$, q^2 respectively and finally letting $b \to \infty$), (3.5) gives

$$S = [-aq; q]_\infty [aq^2; q^2]_\infty \, {}_1\phi_1 \left[\begin{matrix} aq; q^2; aq^3 \\ a^2 q^2 \end{matrix} \right] \qquad ...(3.7)$$

Now, transforming ${}_1\phi_1$ of (3.7) by the special case of the Watson's q-analogue of Whipple's transformation [Bailey[9], 8.5 (2)]

$$ {}_8W_7 \left[a; b, c, e, f, q^{-m}; q; \frac{a^2 q^{2+m}}{bcef}\right] = \frac{[aq, aq/ef; q]_m}{[aq/e, aq/f; q]_m}$$

$$\times \, {}_4\phi_3 \left[\begin{matrix} aq/bc, & e, f, q^{-m}; q; q \\ aq/b, & aq/c, efq^{-m}/a \end{matrix} \right] \qquad ...(3.8)$$

(first replace a by $a^2 q$ then $b = q$, $e = a\sqrt{q}$, $c, f, m \to \infty$ and finally replacing q by q^2), we get (3.2).

Proof of (3.3)—Using (3.1) (with a and b replaced by $-a$ and $a\sqrt{q}$ respectively) the left-hand side of (3.3) becomes

$$\frac{[-aq; q^2]_\infty}{[q; q^2]_\infty} [aq, q; q]_\infty \, {}_2\phi_1 \left[\begin{matrix} -a, 0; q^2; q^2 \\ a^2 q^2 \end{matrix} \right]. \qquad ...(3.9)$$

Transforming the ${}_2\phi_1$ by a special case of (3.8) (by transformation obtained by first letting $b \to \infty$ and then replacing a, e and q by $-a$, b^2q and q^2 respectively), (3.9) becomes

$$[-aq; q^2]_\infty [aq; q]_\infty \, {}_1\phi_1 \left[\begin{matrix} -aq^2; q^2; -aq^2 \\ a^2 q^3 \end{matrix} \right]. \qquad ...(3.10)$$

Next transforming the ${}_1\phi_1$ in (3.10) by a special case of (3.8) [by the transformation obtained from (3.8) by first letting $c, f, m \to \infty$ then replacing a, e, b by $a^2 q, q$ and b respectively and finally replacing q by q^2] yields the transformation (3.3).

Now we are in a position to complete the proof of identities (1.1), (1.4) and (1.6).

Proof of (1.1)—In (1.2) first setting $a = q$ and then $x = -a$ and letting $c, d \to \infty$, we get

$$\sum_{n=0}^{\infty} \frac{[q;q]_{2n}\, q^n}{[q, -aq, -q/a; q]_n} = (1+a) \sum_{n=0}^{\infty} (-a)^n q^{n^2+n}$$

$$- a \frac{[q; q]_\infty}{[-q/a; q]_\infty} \; {}_3\phi_2 \left[\begin{matrix} a\sqrt{q}, -a\sqrt{q}, aq; q; q \\ a^2 q, 0 \end{matrix} \right]. \qquad \text{.. (3.11)}$$

Next, transforming the ${}_3\phi_2$ in the right hand side of (3.11) by (3.2), we get (1.1).

Proof of (1.4)—If S denotes the left-hand side of (1.4) then in view of the transformation (1.3) we get (by letting $a \to 1$, $b, c \to \infty$ in (1.2) and then setting $x = a$)

$$S = \frac{1-a}{1+a} \sum_{n=0}^{\infty} a^n q^{n^2} - \frac{[q, aq; q]_\infty}{[q/a, -a; q]_\infty}$$

$$. {}_3\phi_2 \left[\begin{matrix} a\sqrt{q}, -a\sqrt{q}, -a; q; q \\ a^2 q, 0 \end{matrix} \right] \qquad \text{...(3.12)}$$

Then the proof of (1.4) is completed by transforming the ${}_3\phi_2$ in the right-hand side of (3.12) by (3.3).

Proof of (1.6)— The transformation (1.5) for $a = q$, $b, c \to \infty$ and $x = a$ yields,

$$(1+a) \sum_{n=0}^{\infty} \frac{[q;q]_{2n+1}\, q^{n+1}}{[q;q]_n\, [aq, q/a; q]_{n+1}} + \sum_{n=0}^{\infty} (1 - q^{2n+1})\, a^{n+1} q^{n^2}$$

$$= \frac{[q; q]_\infty\, a}{[1/a; q]_\infty} \; {}_3\phi_2 \left[\begin{matrix} a\sqrt{q}, -a\sqrt{q}, -a; q; q \\ a^2 q, 0 \end{matrix} \right]. \qquad \text{...(3.13)}$$

the proof of (1.6) is now readily completed by transforming the ${}_3\phi_2$ in the right hand side of (3.13) by (3.3).

4. Proofs of (1.9), (1.10) and (1.11)

We began this section by proving (1.9). In view of q-analogue of Saalschütz's theorem [Sears[15], (3.1)], we have

$${}_5\phi_4 \left[\begin{matrix} a^2, b^2, c, d, e; q; q \\ ab\sqrt{q}, -ab\sqrt{q}, f, h \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a^2, b^2; q^2]_n}{[a^2 b^2 q; q]_n} \frac{[c, d, e; q]_n}{[q, f, h; q]_n}$$

(*equation continued on p.* 778)

$$\cdot\, q^{\,n(n-3)/2} \cdot {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{1-n}, q^{1-2n}/a^2 b^2; q^2; q^2 \\ q^{2-2n}/a^2,\ q^{2-2n}/b^2 \end{matrix}\right],$$

provided a, b, c, d or e is of the form q^{-N}, N a non-negative integer.

Writing the series definition for ${}_3\phi_2$ on the right-hand side and diagonalizing the double sum we get

$${}_5\phi_4\left[\begin{matrix} a^2, b^2, c, d, e; q; q \\ ab\sqrt{q}, -ab\sqrt{q}, f, h \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[a^2, b^2; q^2]_n}{[a^2 b^2 q; q^2]_n} \frac{[c, d, e; q]_n}{[q, f, h; q]_n}$$

$$q^{-n(n-3)/2}\ {}_4\phi_3\left[\begin{matrix} q^{-n}, cq^n, dq^n, eq^n; q; q \\ -q, fq^n, hq^n \end{matrix}\right]. \qquad \ldots(4.1)$$

In (4.1) setting $c = q$, $d = x$, $e = -x$, $f = c$, $h = x^2q/c$ we get (1.9) on summing the inner ${}_4\phi_3$ by q-Whipple's theorem[3].

$${}_4\phi_3\left[\begin{matrix} a, q/a, \sqrt{c}, -\sqrt{c}; q: q \\ -q.\ e.\ cq/e \end{matrix}\right] = q^{1/2n(n+1)} \frac{[ea, eq/a, caq/e, cq^2/ae; q^2]_\infty}{[e, cq/e; q]_\infty}$$

where a is of the form q^{-N}. It may be noted that number of interesting summation formulae for Saalschützian ${}_5\phi_4$ series could be deduced from (1.9). For instance in (1.9) setting $b^2 = cq$ and $x = q^{-n}$, we get

$${}_5\phi_4\left[\begin{matrix} a^2, cq, q, q^{-n}, -q^{-n}; q; q \\ aq\sqrt{c}, -aq\sqrt{c}, c, q^{1-2n}/c \end{matrix}\right] = \frac{(1 - cq^{2n})(1 - a^2c)}{(1 - c)(1 - a^2cq^{2n})}$$

whereas for $b^2 = cq$, $a = q^{-n}$ it reduces to

$${}_5\phi_4\left[\begin{matrix} q^{-2n}, cq, q, x, -x; q; q \\ q^{1-n}\sqrt{c}, -q^{1-n}\sqrt{c}, c, x^2 q/c \end{matrix}\right] = \frac{(1 - x^2/c)(c - q^{2n})}{(1 - x^2/c\, q^{2n})(c - 1)}$$

and for $c = q^2$, $a = q^{-n}$, (1.9) yields

$${}_5\phi_4\left[\begin{matrix} q^{-2n}, b^2, q, x, -x; q; q \\ bq^{1/2-n}, -bq^{1/2-n}, q^2, \dfrac{x^2}{q} \end{matrix}\right] = \frac{(1 - q)(b^2 - q^{2n+1})}{(1 - q^{2n+1})(b^2 - q)}$$

which for $q \to 1^-$ reduces to

$${}_4F_3\left[\begin{matrix} -2n, 2b, x, 1; \\ \tfrac{1}{2} + b - n, 2x - 1, 2 \end{matrix}\right] = \frac{1 - 2b + 2n}{(1 + 2n)(1 - 2b)}.$$

To complete the proofs of (1.10) and (1.11) we need the following known transformations.

$${}_{10}W_9\left[a; b_1, c_1, b_2, c_2, b_3, c_3, q^{-n}; q; \frac{a^3 q^{3+n}}{b_1 b_2 b_3 c_1 c_2 c_3}\right] =$$

(*equation continued on p. 779*)

$$\frac{\left[aq,\ \frac{aq}{b_3 c_3};q\right]_n}{\left[\frac{aq}{b_3},\ \frac{aq}{c_3};q\right]_n}\sum_{j,r\geqslant 0}\frac{\left[\frac{aq}{b_1 c_1},\ \frac{aq}{b_2 c_2},\ b_2 c_2;q\right]_j}{[q;q]_r\left[q,\ \frac{aq}{b_1},\ \frac{aq}{c_1};q\right]_j},$$

$$\times\frac{[b_3, c_3, q^{-n}, q]_{r+j}\, q^{r+j}}{\left[\frac{aq}{b_2},\ \frac{aq}{c_2},\ b_3 c_3 q^{-n}/a;q\right]_{r+j}}\left(\frac{aq}{b_2 c_2}\right)^j, \qquad \ldots(4.2)$$

$${}_{10}W_9\left[a;b,x,xq,y,yq,q^{-n},q^{1-n};q^2;\ \frac{a^3 q^{3+2n}}{bx^2 y^2}\right]$$

$$=\frac{\left[aq,\ \frac{aq}{xy};q\right]_n}{\left[\frac{aq}{x},\ \frac{aq}{y};q\right]_n}\ {}_4\phi_5\left[\begin{matrix}x,\ y,\ \sqrt{aq/b)},\ -\sqrt{(aq/b)},\ q^{-n},\ q;\ q\\ \sqrt{(aq)},\ -;\sqrt{(aq)},\ aq/b,\ (xy/a)\,q^{-n}\end{matrix}\right]$$

$$\ldots(4.3)$$

$${}_{10}W_9\left[a;b,x,-x,y,-y,-q^{-n},q^{-n};q;-\frac{a^3 q^{3+2n}}{bx^2 y^2}\right]$$

$$\frac{\left[a^2 q^2,\ \frac{a^2 q^2}{x^2 y^2};q^2\right]_n}{\left[\frac{a^2 q^2}{x^2},\ \frac{a^2 q^2}{y^2};q^2\right]_n}\ {}_5\phi_4\left[\begin{matrix}x^2,\ y^2,\ -aq/b,\ -aq^2/b,\ q^{-2n};\ q^2;\ q^2\\ -aq,\ -aq^2,\ a^2 q^2/b^2,\ x^2 y^2 q^{-2n}/a^2\end{matrix}\right].$$

$$\ldots(4.4)$$

In fact (4.2) is the special case $k = 3$ of generalisation of Watson's q-analogue of Whipples transformation due to Andrews[2] and (4.3) and (4.4) are given in [Verma and Jain[16], (1.3) and (1.4)].

Proof of (1.10)—The LHS of (1.10) is equal to

$$=\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{[x^2;q^2]_n\,(-)^j\,q^{[n(n+1)+j(j+1)2nj/2]}}{[q;q]_j\,[-q,x^2;q]_n}$$

$$+\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{[x^2;q^2]_n\,(-)^n\,q^{[n(n+1)+j(j+1)2nj/2]}}{[q;q]_j\,[-q,-x^2;q]_n}$$

$$=\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{[x^2;q^2]_n\,q^{[n(n+1)+j(j+1)2nj/2]}}{[q;q]_j\,[-q,-x^2;q]_n}\{(-1)^j+(-)^n\}$$

$$=\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{[x^2;q^2]_{n-j}\,q^{n(n+1)/2}\{(-)^j+(-)^{n-j}\}}{[q;q]_j\,[-q,-x^2;q]_{n-j}}$$

(equation continued on p. 780)

$$= 2 \sum_{n=0}^{\infty} q^{n(2n+1)} \sum_{j=0}^{2n} \frac{(-)^j [x^2; q^2]_{2n-j}}{[q; q]_j [-q, -x^2; q]_{2n-j}}$$

$$= 2 \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{[q; q^2]_n} \sum_{j=0}^{n} \frac{(-)^j [x^2; q^2]_j q^{j2}}{[q^2, q^2]_{n-j} [-q^2, -x^2 q; q^2]_j}.$$

(Using (1.9) with $c = - q, b \to \infty$)

$$= 2 \sum_{j=0}^{\infty} \frac{[x^2; q^2]_j (-)^j q^{3j^2+j}}{[-q^2, -x^2 q, q; q^2]_j} \sum_{r=0}^{\infty} \frac{q^{2r^2+4rj+r}}{[q^2, q^{1+2j}; q^2]_r}$$

$$= 2 \sum_{j=0}^{\infty} \frac{[x^2; q^2]_j (-)^j q^{3j^2+j}}{[-q^2, -x^2 q, q; q^2]_j [q^{1+2j}; q^2]_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(-)^n q^{n^2+2nj} [q^2; q^2]_{n+j}}{[q^2; q^2]_n [q^2; q^2]_j}$$

(using last equation on page 38 of Andrews[4])

$$= \frac{2}{[q; q^2]_\infty} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q^2; q^2]_{n+j} [x^2; q^2]_j (-)^{n+j} q^{n^2+2nj+3j^2+j}}{[q^2, -q^2, -x^2 q; q^2]_j [q^2; q^2]_n}$$

$$= 2 [-q; q]_\infty \sum_{r=0}^{\infty} (1 - q^{4r+1}) \frac{[-q, -q^2/x^2; q^2]_r}{[-q^2, -x^2 q; q^2]_r} x^{2r} q^{4r^2}$$

(using (4.2) with first $q \to q^2$ then $a = q, c_3 = q^2, b_1 = q, c_1 = - q^2/x^2$,

$b_2, c_2, b_3, n \to \infty$)

which reduces to the right-hand side of (1.10) on using (4.3) (with $a = -q, x=q, b= -\frac{q^2}{x^2}, n \to \infty$ and then q replaced by $-q$).

Proof of (1.11)—The left-hand-side of (1.11)

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[x^2; q^2]_{n+1} \{(-)^r - (-)^n\} q^{[r(r+1)+n(n+1)+2rn/2]}}{[-q; q]_n [-x^2 q; q]_{n+1} [q; q]_r}$$

$$= \sum_{m=0}^{\infty} q^{m(m+1)/2} \sum_{r=0}^{m} \frac{[x^2; q^2]_{m+1-r} \{(-)^r - (-)^{m-r}\}}{[-q; q]_{m-r} [-x^2 q; q]_{m+1-r} [q; q]_r}$$

(*equation continued on p.* 781)

$$= 2 \sum_{m=0}^{\infty} q^{(m+1)(2m+1)} \sum_{r=0}^{2m+1} \frac{[x^2; q^2]_{2m+2-r} (-)^r}{[-q; q]_{2m+1-r} [-x^2 q; q]_{2m+2-r} [q; q]_r}$$

$$= 4 \sum_{m=0}^{\infty} \frac{q^{(m+1)(2m+1)}}{[q;q^2]_{m+1}} \sum_{r=0}^{m} \frac{[x^2; q^2]_{r+1} (-)^{r+1} q^{(r+1)s^2}}{[-q, -x^2q^2; q^2]_{r+1} [q^2;q^2]_{m-r}}$$

(using (1.9) with $a = q^{-2m-2}$, $c = -1$, $b \to \infty$)

$$= 4\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{[x^2; q^2]_{r+1} (-)^{r+1} q^{3r^2+2s^2+4rs+5r+3s+2}}{[q^2; q^2]_s [-q, -x^2 q^2; q^2]_{r+1} [q; q^2]_{r+s+1}}$$

$$= 4\sum_{r=0}^{\infty} \frac{[x^2; q^2]_{r+1} (-)^{r+1} q^{3r^2+5r+2}}{[q, -x^2 q^2, -q; q^2]_{r+1}} \sum_{s=0}^{\infty} \frac{q^{2s^2+4rs+3s}}{[q^2, q^{3+2r}; q^2]_s}$$

$$= 4\sum_{r=0}^{\infty} \frac{[x^2; q^2]_{r+1} (-)^{r+1} q^{3r^2+5r+2}}{[-x^2 q^2, q, -q; q^2]_{r+1} [q^{3+2r}; q^2]_{\infty}}$$

$$\cdot \sum_{n=0}^{\infty} \frac{[q^{2+2r}; q^2]_n (-)^n q^{n^2+2nr+2n}}{[q^2; q^2]_n}$$

$$= \frac{4}{[q; q^2]_{\infty}} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[x^2; q^2]_{r+1} [q^2; q^2]_{n+r} (-)^{n+r+1} q^{n^2+3r^2+2nr+5r+2n+2}}{[q^2; q^2]_r [q^2; q^2]_n [-x^2 q^2, -q; q^2]_{r+1}}$$

$$= 4 [-q; q]_{\infty} \frac{(q^5 - q^2)(1 - x^2)}{(1 + q)(1 + x^2q^2)}$$

$$\times \lim_{b_2, c_2 b_3, n \to \infty} {}_{10}W_9 \left[q^3; -q^2, -\frac{q}{x^2}, b_2, c_2, b_3, q^2, q^{-2n}; q^2; \frac{x^2 q^{10+2n}}{b_2 c_2 b_3}\right]$$

(using (4.2))

$$= [-q; q]_{\infty} \frac{(q^5 - q^2)(1 - x^2)}{(1 + q)(1 + x^2 q^2)(1 - q^6)}$$

$$\cdot \sum_{n=0}^{\infty} \frac{[x^2 q^4; q^2]_{2n} (-)^n q^{2n^2+4n}}{[-q^5; q^2]_{2n} [x^4 q^8; q^4]_n}.$$

Lastly using (4.4) with $q \to q^2$ and then $a = q^3$, $x = q^2$, $b = -\frac{q}{x^2}\,y$, $n \to \infty$ the above expression reduces to the RHS of (1.11).

5. Two Interesting Quadratic Transformations

In this section we prove the following quadratic transformations

$$
{}_6\phi_5\left[\begin{matrix} a^2,\ -aq,\ x,\ y,\ ac^2q^{1+m}/xy,\ q^{-m};\ q\ ;q \\ -a,\ acq/x,\ acq/y,\ xyq^{-m}/c,\ acq^{1+m} \end{matrix}\right]
$$
$$
= \frac{[cq/x,\ cq/y,\ acq,\ acq/xy;\ q]_m}{[cq,\ acq/x,\ acq/y,\ cq/xy;\ q]_m}
$$
$$
\times\ {}_{12}W_{11}\left[c;\ a,\ x,\ y,\ \sqrt{(c/a)},\ -\sqrt{(c/a)},\ \sqrt{cq/a)},-\sqrt{(cq/a)}\right.
$$
$$
\left.,\ \frac{ac^2\,q^{1+m}}{xy},\ q^{-m},\ q;\ q^2\right] \qquad \ldots(5.1)
$$

$$
{}_{12}W_{11}\left[-\ a;\ b,\ q\sqrt{a},\ -q\sqrt{a},\ x,\ -x,\ y,\ -\ y,\ z,\ -z;\ q;\ a^3\,q^2/bx^2\,y^2\,z^2\right]
$$
$$
= \frac{\left[a^2\,q^2,\ \frac{a^2\,q^2}{x^2\,y^2},\ \frac{a^2\,q^2}{x^2\,z^2},\ \frac{a^2\,q^2}{y^2\,z^2};\ q^2\right]_\infty}{\left[\frac{a^2\,q^2}{x^2},\ \frac{a^2\,q^2}{y^2},\ \frac{a^2\,q^2}{z^2},\ \frac{a^2\,q^3}{x^2y^2z^2};q^2\right]_\infty}
$$
$$
\cdot\,{}_5\phi_4\left[\begin{matrix} x^2,\ y^2,\ z^2,\ a/b,\ aq/b;\ q^2;\ q^2 \\ a,\ aq,\ a^2q^2/b^2,\ x^2\,y^2\,z^2/a^2 \end{matrix}\right]
$$
$$
+ \frac{[a^2\,q^2,\ x^2,\ y^2,\ z^2,\ a^2q^4/b^2\,x^2\,y^2\,z^2,\ a^4\,q^4/x^2\,y^2\,z^2;\ q^2]_\infty}{\left[\frac{a^2q^2}{x^2},\ \frac{a^2\,q^2}{y^2},\ \frac{a^2q^2}{z^2},\ \frac{a^2\,q^2}{b^2},\ \frac{x^2\,y^2\,z^2}{a^2\,q^2},\ \frac{a^2\,q^4}{x^2y^2\,z^2};\ q^2\right]_\infty}
$$
$$
\times \frac{\left[\frac{a}{b},\ \frac{a^3\,q^2}{x^2\,y^2\,z^2};\ q\right]_\infty}{\left[a,\ \frac{a^3\,q^2}{bx^2y^2z^2}\ ;q\right]_\infty}
$$
$$
\cdot\,{}_5\phi_4\left[\begin{matrix} \frac{a^2\,q^2}{x^2\,y^2},\ \frac{a^2\,q^2}{x^2\,z^2},\ \frac{a^2\,q^2}{y^2\,z^2},\ \frac{a^3\,q^2}{bx^2\,y^2z^2},\ \frac{a^3\,q^3}{bx^2\,y^2\,z^2};\ q^2;q^2 \\ \frac{a^4\,q^4}{x^2y^2z^2},\frac{a^2\,q^4}{b^2x^2y^2z^2},\ \frac{a^3\,q^2}{x^2\,y^2\,z^2},\ \frac{a^3\,q^3}{x^2\,y^2\,z^2} \end{matrix}\right]. \qquad \ldots(5.2)
$$

The transformation (5.1) is a q-analogue of the following interesting transformation of 'half-poised' hypergeometric series [see Whipple[17], p. 70] for definition of 'half-poised' series)

$$
{}_5F_4\left[\begin{matrix} 2a,\ 1+a,\ x,\ y,\ -m; \\ a,1+a+c-x,\ 1+a+c-y,\ 1+a+c+m \end{matrix}\right] =
$$

(equation continued on p. 783)

$$= \frac{(1+a+c)_m\,(1+a+c-x-y)_m}{(1+a+c-x)_m\,(1+a+c-y)_m}$$

$${}_5F_4\left[\begin{matrix} x,\ y,\ (c-a)/2,\ (1+c-a)/2,\ -m; \\ (1+c-a,\ x+y-a-c-m,\ (2+c+a)/2,\ (1+c+a)/2 \end{matrix}\right].$$

Proof of (5.1)—Specializing the parameters in the transformation [Sears[14], (8.3)], we get

$${}_4\phi_3\left[\begin{matrix} a^2-aq,\ c,\ q^{-n};\ q;\ q, \\ -a,\ aq^{1-n},\ acq \end{matrix}\right] = [a,\ c/a,\ aq/c;\ q]_n/acq,\ [a/c;\ q]_n.$$

Now replacing c by cq^n, multiplying both sides by

$$\frac{[1/a,\ c,\ x,\ y,\ ac^2\,q^{1+m}/xy,\ q^{-m};\ q]_n\,(1-cq^{2n})\,q^n}{[q,\ acq,\ cq/x,\ cq/y,\ xyq^{-m}/ac, cq^{1+m};\ q]_n\,(1-c)}$$

and summing with respect to n from $n=0$ to m, we get (5.1) on interchanging the order of summations on the left hand side and summing the inner ${}_8W_7$ by [Bailey[9]; 8.3 (1)].

Proof of (6.2)—In the non-terminating version of Saalschütz' summation formula [Sears[15], (5.1)];

$$\frac{[aq,\ aq/fg,\ ag/fh,\ aq/gh;\ q]_\infty}{[f,\ g,\ h,\ aq/fgh;\ q]_\infty}\ {}_3\phi_2\left[\begin{matrix} f,\ g,\ h;\ q;\ q \\ aq,\ fgh/a \end{matrix}\right]$$

$$+\ \frac{[a^2\,q^2/fgh;\ q]_\infty}{[fgh\,/aq:\ q]_\infty}\ {}_3\phi_2\left[\begin{matrix} aq/gh,\ aq/fh,\ aq/fg;\ q:\ q \\ aq^2/fgh,\ a^2q^2/fgh \end{matrix}\right]$$

$$= \frac{[aq/f,\ aq/g,\ aq/h;q]_\infty}{[f,\ g,\ h;\ q]_\infty}. \qquad \text{...(5.3)}$$

First replacing q by q^2 and then a, f, g, h by $a^2\,q^{4r},\ x^2\,q^{2r},\ y^2\,q^{2r},\ z^2\,q^{2r}$ respectively and multiplying by

$$\frac{[-a,\ b;\ q]_r\,(1-a^2\,q^{4r})}{[q,\ -aq/b;\ q]_r\,(1-a^2)}\ (-\,a^3\,q^2/bx^2\,y^2\,z^2)^r$$

and summing with respect to r from 0 to ∞, we get

$$\frac{\left[\frac{a^2\,q^2}{x^2},\ \frac{a^2\,q^2}{y^2},\ \frac{a^2\,q^2}{z^2};\ q^2\right]_\infty}{[x^2,\ y^2,\ z^2;\ q^2]_\infty}\ {}_{12}W_{11}\Bigg[-\,a;\ b,\ q\sqrt{a},\ -q\sqrt{a},\ x,$$

$$,-\,x,\ y,\ -\,y,\ z,\ -\,z;\ q;\ \frac{a^3\,q^2}{bx^2\,y^2\,z^2}\Bigg]$$

$$= \frac{[a^2\,q^2,\ a^2\,q^2/x^2\,y^2,\ a^2\,q^2\,/x^2\,z^2,\ a^2\,q^2/y^2\,z^2,\ q^2]_\infty}{[x^2,\ y^2,\ z^2,\ a^2\,q^2/x^2\,y^2\,z^2;\ q^2]_\infty}$$

(equation continued on p. 784)

$$\times \sum_{n=0}^{\infty} \frac{[x^2, y^2, z^2; q^2]_n\, q^{2n}}{[q^2, a^2 q^2, x^2 y^2 z^2/a^2; q^2]_n}$$

$$\times\ {}_8W_7\,[-a; b, q\sqrt{a}, -q\sqrt{a}, -q^{-n}, q^{-n}; q; aq^{2n}/b]$$

$$+ \frac{[a^4 q^4/x^2 y^2 z^2; q^2]_\infty}{[x^2 y^2 z^2/a^2 q^2; q^2]_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{[a^2 q^2/x^2 y^2, a^2 q^2/x^2 z^2, a^2 q^2/y^2 z^2; q^2]_n\, q^{2n}}{[q^2, a^2 q^4/x^2 y^2 z^2, a^4 q^4/x^2 y^2 z^2; q^2]_n}$$

$$\times\ {}_8W_7\,[-a, b, q\sqrt{a}, -q\sqrt{a}, \frac{x y z q^{-n-1}}{a},$$

$$, \frac{-x y z q^{-n-1}}{a}; q; \frac{a^2 q^{2+2n}}{bx^2 y^2 z^2}]. \qquad \ldots(5.4)$$

Using the summation

$${}_8W_7\,[-a; b; -q\sqrt{a}, q\sqrt{a}, -z, z; q; a/bz^2]$$

$$= \frac{[a^2 q^2, a^2 q^2/b^2 z^2; q^2]_\infty\, [a/b, a/z^2; q]_\infty}{[a^2 q^2/b^2, a^2 q^2/z^2; q^2]_\infty\, [a, a/bz^2; q]_\infty} \qquad \ldots(5.5)$$

to sum both the ${}_8W_7$ in (5.4) we get (5.2). The proof of (5.5) is completed by first replacing *a by* $-a$ in the transformotion Verma and Jain[16] (5.1) and then setting $x^2 = aq$, $y^2 = aq^2$ and using (5.3) to evaluate the right hand side.

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UNSTEADY LAMINAR BOUNDARY LAYER FORCED FLOW OVER A MOVING WALL WITH A MAGNETIC FIELD

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The unsteady laminar incompressible two-dimensional and axisymmetric stagnation-point flows over a moving wall with a magnetic field have been studied when the free stream velocity and the wall velocity vary arbitrarily with time. It has been shown that self-similar solution is possible when the free stream velocity, the wall velocity and square of the magnetic field vary inversely as a linear function of time. The partial differential equations governing the semi-similar case and the ordinary differential equations governing the self-similar case have been solved numerically using the finite-difference scheme in combination with the quasilinearization technique. Analytical solutions have also been obtained for certain limiting cases. The skin friction and heat transfer are appreciably affected by the free stream velocity distribution, magnetic field and wall velocity. However, their effects on the heat transsfer is comparatively less compared to the skin friction.

1. Introduction

Flows over moving walls are of interest in a number of technical applications, especially in metallurgy and chemical processes industries. Such flows belong to a separate class of problems of boundary layer theory which is distinct from those over stationary bodies. Sakiadis[1] was probably the first to study the flow over a moving boundary in a fluid at rest. Subsequently, several investigators[2-8] considered the behaviour of boundary layer on moving surfaces under different situations. All these studies pertain to steady flows. The unsteady flow over a moving wall with forced flow has not been studied so far. However, the unsteady forced flow over a stationary wall has been studied by Yang[9] when the free stream velocity varies inversely as a linear function of time. Also, the unsteady flow over a moving wall in a fluid at rest has been studied recently by Surma Devi and Nath[10].

The aim of the present analysis is to study the unsteady laminar incompressible forced flow over a moving boundary with an applied magnetic field when the free stream velocity and the wall velocity vary arbitrarily with time. It has also been shown that the self-similar solution is possible when the free stream velocity, wall velocity and the square of the magnetic field vary inversely as a linear function of time. It may be noted that here the wall is not moving as a rigid boundary as considered by Sakiadis[1], but it is stretched. The partial differential equations governing the semi-similar case and the ordinary differential equations governing the self-similar case have been solved numerically using a finite-difference scheme in combination with the quasilinearization technique[11,12]. Also analytical solutions of certain limiting cases have been obtained. The results have been compared with those available in the literature.

2. Governing Equations

We consider that a two-dimensional or an axisymmetric body is moving with time-dependent velocity u_w in a laminar incompressible fluid with free stream velocity u_e which also varies with time (see inset of Fig. 1). The fluid is assumed to be electrically conducting and a magnetic field B fixed relative to the fluid is applied in the dire-

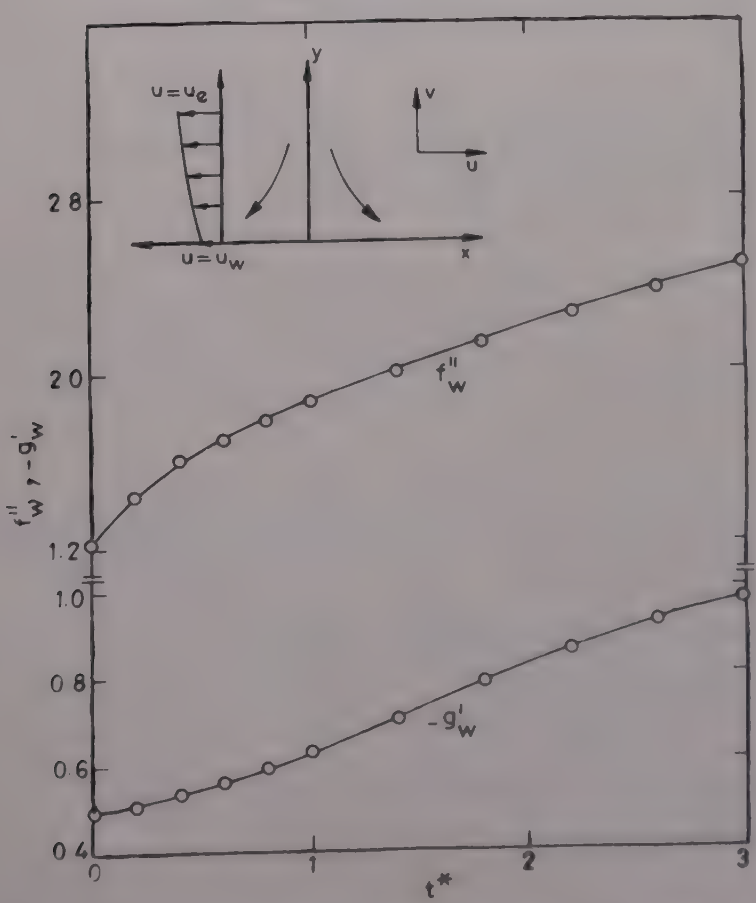

Fig. 1. Comparison of skin-friction and heat-transfer results $\left(f''_w, -g'_w\right)$ for $\varphi(t^*) = 1 + t^*$, $M = b = j = 0$, $Pr = 0.72$, ———, Present results; 0, Kumari and Nath.

ction perpendicular to the body. The magnetic Reynolds number is assumed to be small, hence the induced magnetic field will be small compared to the applied magnetic field and can be neglected. Since we are interested in the stagnation point region, the viscous dissipation and Joule heating terms are neglected as they are small in the neighbourhood of the stagnation point. The Hall effect is also neglected. The wall and free srteam temperatures are taken to be constants. By assuming that Prandtl's boundary layers assumptions are valid in the present case, the governing equations can be expressed as

$$(r^j u)_x + (r^j v)_y = 0 \qquad \ldots(1a)$$

$$u_t + uu_x + vu_y = (u_e)_t + u_e (u_e)_x + \nu u_{yy} + (\sigma B^2 u_e/\rho)(1 - u/u_e) \qquad \ldots(1b)$$

$$T_t + uT_x + vT_y = Pr^{-1} \nu T_{yy}. \qquad \ldots(1c)$$

The initial and boundary conditions are given by

$$u(x, y, 0) = u_i(x, y),\ v(x, y, 0) = v_i(x, y),\ T(x, y, 0) = T_i(x, y) \qquad \ldots(2a)$$

$$\left.\begin{aligned} &u(x, 0, t) = u_w(x, t),\ v(x, 0, t) = 0,\ T(x, 0, t) = T_w, \\ &u(x, \infty, t) = u_e(x, t),\ T(x, \infty, t) = T_\infty . \end{aligned}\right\} \qquad \ldots(2b)$$

2.1 *Semi-similar Equations*

In order to reduce the number of independent variables from 3 to 2 in eqns. (1a) to (1c), we apply the following transformations

$$\left.\begin{aligned} &\eta = (1 + j)^{1/2} (a/\nu)^{1/2} y,\ u = ax\,\varphi(t^*)\, f'(\eta, t^*),\ t^* = at, \\ &v = -(1 + j)^{1/2} (a\nu)^{1/2} \varphi(t^*) f(\eta, t^*),\ u_w = a_1 x\,\varphi(t^*) \end{aligned}\right\} \qquad \ldots(3a)$$

$$\left.\begin{aligned} &u_e = ax\,\varphi(t^*),\ (T - T_\infty)/(T_w - T_\infty) = g(\eta, t^*),\ r \approx x,\ M = Ha^2/\mathrm{Re}_L \\ &Ha^2 = \sigma B^2 L^2/\mu,\ \mathrm{Re}_L = aL^2/\nu,\ b = u_w/u_e \end{aligned}\right\} \qquad \ldots(3b)$$

to eqns. (1a) to (1c), we find that eqn. (1a) is satisfied identically and eqns. (1b) to (1c) reduce to

$$f''' + \varphi f f'' + (1 + j)^{-1} [\varphi (1 - f'^2) + \varphi^{-1} \varphi_{t^*} (1 - f') - f'_{t^*} + M(1 - f')] = 0. \qquad \ldots(4)$$

$$Pr^{-1} g'' + \varphi f g' - (1 + j)^{-1} g_{t^*} = 0. \qquad \ldots(5)$$

The boundary conditions are given by

$$\left.\begin{aligned} &f = 0,\ f' = b,\ g = 1 \text{ at } \eta = 0, \\ &f' \to 1,\ g \to 0 \text{ as } \eta \to \infty \end{aligned}\right\} \text{ for } t^* \geqslant 0. \qquad \ldots(6)$$

The flow is initially assumed to be steady ($t^* = 0$) and then changes to unsteady state ($t^* > 0$). Therefore, the initial conditions are given by the steady-state equations obtained by putting

$$t^* = 0, \varphi = 1, f'_{t^*} = g_{t^*} = \varphi_{t^*} = 0 \qquad \ldots(7)$$

in eqns. (4) and (5) and the steady-state equations are

$$f''' + ff'' + (1 + j)^{-1} (1 - f'^2) + M (1 - f') = 0 \qquad \ldots(8)$$

$$Pr^{-1} g'' + fg' = 0. \qquad \ldots(9)$$

It may be noted that eqns. (4) and (5) contain two independent variables and are known as semi-similar equations.

Here x and y are distances along and perpendicular to the surface, respectively; u and v are the components of velocity along x and y directions, respectively: t and t^* are the dimensional and dimensionless times, respectively; T is the temperature: μ and ν are the coefficient of viscosity and kinematic viscosity, respectively; η is the similarity variable; f is the dimensionless stream function; f', g are the dimensionless velocity and temperature, respectively; B is the magnetic field; a is the velocity gradient at $t^* = 0$, r is the radius of the axisymmetric body; $j = 0$ and 1 for two-dimensional and axisymmetric flows, respectively; M, Ha and Re_L are the magnetic parameter, Hartmann number and Reynolds number, respectively; a_i is the gradient of the wall velocity at $t^* = 0$; b is the ratio of the velocity of the wall and the free stream velocity ($b \gtrless 0$ according to whether the velocities of the wall and free stream are in the same direction or in opposite direction); σ, Pr and L are respectively, electrical conductivity, Prandtl number and charcteristic length. φ is an arbitrary function of t^* having continuous first derivative for $t^* \geqslant 0$. The subscript i denotes the initial conditions; the subscripts e, w, and ∞ denote conditions at the edge of the boundary layer, on the wall and in the free stream, respectively; the subscripts t, t^*, x and y, denote partial derivatives with respect to t, t^*, x and y, respectively; and prime denotes derivatives with respect to η.

It may be noted that eqns. (4) and (5) for $b = M = 0$ and $j = 0$ and 1 reduce to eqns. (3a) and (3c) of Kumari and Nath[13] with $c = 0$ and 1 ($c = 0$ for a two-dimensional flow and $c = 1$ for an axisymmetric flow). Our eqns. (4) and (5) for $j = 0$ (two-dimensional flow) are identical to eqns. (3a) and (3c) of Kumari and Nath[13] with $c = 0$. However, our eqns. (4) and (5) for $j = 1$ (axisymmetric flow) differ from eqns. (3a) and (3c) of Kumari and Nath[13] with $c = 1$ by a scaling factor.

The skin friction and heat transfer coefficients can be expressed as

$$\left.\begin{aligned} C_f &= 2\tau_w/[\rho\ (u_e^2\ )_{t^*=0}] = 2\ (Re_x)^{-1/2}\ \varphi\ (t^*)\ f''_w\ , \\ Nu &= x\ (\partial T/\partial y)_w/(T_w - T_\infty) = -\ (Re_x)^{1/2}\ g'_w \end{aligned}\right\} \qquad \ldots(10a)$$

where

$$\tau_w = \mu\,(\partial u/\partial y)_w,\ \mathrm{Re}_x = ax^2/\nu,\ (u_e)_{t^*=0} = ax. \qquad \ldots(10b)$$

Here C_f and Nu are the surface skin friction coefficient and Nusselt number (heat transfer coefficients), respectively; τ_w is the shear stress at the wall; f''_w and $-g'_w$ are the skin friction and heat transfer parameters at the wall; Re_x is the local Reynolds number; and ρ is the density.

2.2. *Self-similar Equations*

The set of eqns. (1a) to (1c) is partial differential equations with three independent variables. It can be shown that if the free stream velocity and the wall velocity vary inversely as a linear function of time and directly as a linear function of x (i. e. $u_e = ax\,(1-\lambda t^*)^{-1}$, $u_w = a_1\,x\,(1-\lambda t^*)^{-1}$), and the magnetic field as a square root of the linear function of time then eqns. (1a) to (1c) admit self-simllar solutions i. e. they are reduced to a set of ordinary differential equations. We apply the following transformations

$$\left.\begin{aligned}
&\eta = (1+j)^{1/2}\,(a/\nu)^{1/2}\,(1-\lambda t^*)^{-1/2}y,\ t^* = at,\ \lambda t^* < 1,\\
&u = ax\,(1-\lambda t^*)^{-1}f'(\eta),\ v = -(1+j)^{1/2}\,(1-\lambda t^*)^{-1/2}f(\eta),\\
&(T-T_\infty)/(T_w-T_\infty) = g(\eta),\ B = B_0\,(1-\lambda t^*)^{-1/2},\ M = Ha^2/\mathrm{Re}_L,\\
&Ha^2 = \sigma B_0^2\,L^2/\mu,\ u_e = ax\,(1-\lambda t^*)^{-1},\ u_w = a_1\,x\,(1-\lambda t^*)^{-1}.
\end{aligned}\right\} \qquad \ldots(11)$$

to eqns. (1a) to (1c) and we find that eqn. (1a) is satisfied identically and eqns. (1b) to (1c) reduce to

$$f''' + ff'' + (1+j)^{-1}\,[(1-f'^2) + \lambda\,(1-f'-\eta f''/2) + M\,(1-f')] = 0. \qquad \ldots(12)$$

$$Pr^{-1}\,g'' + fg' - (1+j)^{-1}\,\lambda\eta\,g'/2 = 0. \qquad \ldots(13)$$

The boundary conditions are

$$f = 0,\ f' = b,\ g = 1 \text{ at } \eta = 0;\ f' - 1 = g = 0 \text{ as } \eta \to \infty. \qquad \ldots(14)$$

Here λ is the parameter characterizing the unsteadiness in the flow field and B_0 is the value of the magnetic field B at $t^* = 0$. $\lambda >< 0$ according as the flow is accelerating or decelerating. Also the magnetic field B is assumed to vary as the square root of a linear function of time as given in (11) in order to obtain self-similar solution. In actual practice, it may not be possible to create and maintain such a magnetic field. Inspite of this weakness, the results may be used to gain scme insight into the characteristics of flow based on more realistic distribution of the magnetic field.

It may be noted that the governing self-similar equations (12) and (13) with boundary conditions (14) reduce to those of Yang[9] for $M = b = j = 0$. The self-similar solution implies that the solution at one value of time t is similar to the solution at any other value of time t. The advantage of the similarity solution is that the partial differential equations reduce to ordinary differential which is a great mathematical simplification. However, there are certain limitations to such solutions. The partial differential equations do not impart their parabolic nature to the ordinary differential equations on whose solution it is not possible to impose an arbitrary condition at an initial time $t = t_0$, because the solutions at all values of time t become equivalent. Therefore, the self-similar solution can be considered as asymptotic and will be correct in some limit $t \rightarrow t_1$.

The skin-friction and heat-transfer coefficients are given by

$$\left.\begin{aligned} C_f &= 2\,\tau_w/(\rho u_e^2) = 2\,(1+j)^{1/2}\,(\overline{Re}_x)^{-1/2}\,f''_w,\ \overline{Re}_x = u_e\,x/\nu \\ Nu &= xT_y/(T_w - T_\infty) = (1+j)^{1/2}\,(\overline{Re}_x)^{1/2}\ g'_w . \end{aligned}\right\} \quad \text{...(15)}$$

3. Solution of Governing Equations

3.1. *Asymptotic Solution*

In this section, we consider the asymptotic behaviour of the governing equations (12) and (13) (i. e. the behaviour of the equations when η becomes large). This will enable us to find the range of values of λ for which similarity solution is valid. For large η following the analysis of Watson and Wang[15], we set

$$f(\eta) = \eta + f_1(\eta),\ g(\eta) = g_1(\eta). \quad \text{...(16a)}$$

From boundary conditions (14), it is evident that

$$f_1 \rightarrow 0,\ f_1' \rightarrow 0,\ g_1 \rightarrow 0 \text{ as } \eta \rightarrow \infty \quad \text{...(16b)}$$

where f_1 and g_1 are small. Now linearizing equations (12) and (13) using relations given in eqn. (16a) and integrating the resulting equation corresponding to eqn. (12) once and using the appropriate boundary conditions, we get

$$f_1'' + \alpha\eta f_1' - [(1+j)^{-1}(2+\lambda+M) + \alpha] f_1 = 0 \quad \text{...(17a)}$$

$$Pr^{-1} g'' + \alpha\eta g_1' = 0,\ \alpha = 1 - 2^{-1}(1+j)^{-1}\lambda. \quad \text{...(17b)}$$

The solution of eqn. (17b) satisfying the relevant boundary condition given in (16b) can be written in the form

$$g = g_1 = -A \int_\eta^\infty \exp(-Pr\,\alpha\,\eta^2/2)\,d\eta \quad \text{..(18)}$$

where A is a constant. We apply the following transformation to equation (17a)

$$f_1 = \exp(-\alpha\,\eta^3/4)\;H. \qquad \ldots(19)$$

Consequently, equation (17a) can be reduded to

$$H'' - [3/2 + (1+j)^{-1}(2 + \lambda/4 + M) + \alpha^2\,\eta^2/4]\,H = 0 \qquad \ldots(20)$$

where $H \to 0$ as $\eta \to \infty$. Equation (20) is Weber's type of equation whose solution for large η can be written in terms of parabolic cylinder functions as[14]

$$H = A_1 \exp(-\alpha^2\,\eta^2/4)\,(\alpha\eta)^n\,P_1(\eta) + B_1 \exp(\alpha^2\,\eta^2/4)\,(\alpha\eta)^{-n-1}\,P_2(\eta) \qquad \ldots(21)$$

where

$$\left.\begin{aligned} P_1(\eta) &= [1 - 2^{-1}\,n\,(n-1)\,(\alpha\eta)^{-2} + O\,(\alpha\eta)^{-4} -, \ldots],\\ P_2(\eta) &= [1 + 2^{-1}\,n\,(n+1)\,(\alpha\eta)^{-2} + O\,(\alpha\eta)^{-4} +, \ldots],\\ n &= -2 - (1+j)^{-1}(2 + \lambda/4 + M). \end{aligned}\right\} \qquad \ldots(22)$$

Since H tends to zero as $\eta \to \infty$, the divergent part of the solution of H i. e. $\exp(\alpha^2\eta^2/4)$ will be omitted. Hence from equations (19) aud (21) we find that

$$f_1 = A_1 \exp[-\alpha\,(\alpha+1)\,\eta^2/4]\,(\alpha\,\eta)^n\,P_1(\eta). \qquad \ldots(23)$$

It is clear from eqns. (18) and (23) that both g or g_1 and f_1 decay to zero exponentially if $\alpha > 0$ i. e. $\lambda < 2\,(1+j)$. This fixes the upper limit of λ. The lower limit of λ is given by that value of λ ($\lambda < 0$) for which the skin friction parameter f''_w vanishes.

3.2. *Analytical Solution*

It may be remarked that it is not possible to obtain closed form solutions of eqns. (12) and (13) under conditions (14). However if $b = 1$, closed form solutions of eqns. (12) and (13) satisfying conditions (14) can be obtained and they are expressed as

$$f = \eta$$

$$g = 1 - 2\,(\alpha_1/\pi)^{1/2} \int_0^{\eta} \exp(-\alpha_1\eta^2)\,d\eta \qquad \ldots(24b)$$

$$\alpha_1 = (Pr/2)\,[1 - 2^{-1}\,(1+j)^{-1}\,\lambda]. \qquad \ldots(24c)$$

Since we are interested in the closed form solution of eqn. (12) in the neighbourhood of $b = 1$, we perturb f as

$$f = \eta + \epsilon f_2(\eta) + O(\epsilon^2),\ \epsilon = 1 - b \qquad \ldots(25)$$

Linearizing eqn. (12) with the help of (25), we get

$$f_2''' + \alpha\eta\,f_2'' - (1+j)^{-1}(2 + \lambda + M)\,f_2' = 0. \qquad \ldots(26)$$

The appropriate boundary conditions are

$$f_2(0) = 0, \; f_2'(0) = -1, \; f_2'(\infty) = 0. \qquad \ldots(27)$$

We apply the following transformation

$$f_2'(\eta) = \exp(-\alpha\eta^2/4) \; F(\eta) \qquad \ldots(28)$$

to eqn. (26) which then reduces to

$$F'' - [(\alpha/2) + (1+j)^{-1}(2 + \lambda + M) + \alpha^2 \eta^2/4] F = 0. \qquad \ldots(29)$$

The relevant boundary conditions are

$$F(0) = -1, \; F(\infty) = 0. \qquad \ldots(30)$$

It may be noted that eqn. (29) is Weber's type of equation and its solution under condition (30) can be expressed as

$$F = -\exp(-\alpha^2\eta^2/4) \left[{}_1F_1\left(-\frac{a}{2}, \frac{1}{2}, \frac{\alpha^2\eta^2}{2}\right) + (B_2/A_2)\,\alpha\eta\; {}_1F_1\left(\frac{1-a}{2}, \frac{3}{2}, \frac{\alpha^2\eta^2}{2}\right)\right]. \qquad \ldots(31)$$

From eqns. (28) and (31), we get

$$f_2' = -\exp[-\alpha(\alpha+1)\eta^2/4] \left[{}_1F_1\left(-\frac{a}{2}, \frac{1}{2}, \frac{\alpha^2\eta^2}{2}\right) + (B_2/A_2)\,\alpha\,\eta\; {}_1F_1\left(\frac{1-a}{2}, \frac{3}{2}, \frac{\alpha^2\eta^2}{2}\right)\right] \qquad \ldots(32)$$

where

$$\left.\begin{aligned} {}_1F_1\left(-\frac{a}{2}, \frac{1}{2}, \frac{\alpha^2\eta^2}{2}\right) &= \left[1 - \frac{a}{2}\,\frac{\alpha^2\eta^2}{2} + \frac{\frac{a}{2}\left(\frac{a}{2}-1\right)}{3\,L\,2} \times \left(\frac{\alpha^2\eta^2}{2}\right)^2 - \ldots\right] \\ {}_1F_1\left(\frac{1-a}{2}, \frac{3}{2}, \frac{\alpha^2\eta^2}{2}\right) &= \left[1 + \frac{1-a}{3}\,\frac{\alpha^2\eta^2}{2} + \frac{(1-a)(3-a)}{3.\,5.\,2^3}(\alpha^2\eta^2)^2 + \ldots\right] \end{aligned}\right\} \qquad \ldots(33a)$$

$$\left.\begin{array}{l} a = -(\alpha + 1)/2 - (1 + j)^{-1}(2 + \lambda + M),\ \alpha = [1 - 2^{-1}(1 + j)^{-1}\lambda], \\ A_2 = \Gamma(1/2)/\Gamma\left(\dfrac{1 - a}{2}\right),\ B_2 = 2^{-1/2}\,\Gamma(-1/2)/\Gamma(-a/2). \end{array}\right\} \quad \ldots(33b)$$

The shear stress at the wall is given by

$$f_2''(0) = \frac{2^{1/2}\alpha\,\Gamma\{\frac{1}{4}(\alpha + 3) + \frac{1}{2}(1 + j)^{-1}(2 + \lambda + M)\}}{\Gamma\{\frac{1}{4}(\alpha + 1) + \frac{1}{2}(1 + j)^{-1}(2 + \lambda + M)\}}\,. \quad \ldots(34)$$

3.3. *Numerical Solution*

The partial differential equations (4) and (5) under boundary conditions (6) and initial conditons (8) and (9) and the ordinary differential equations (12) and (13) under boundary conditions (14) have been solved numerically using an implicit finite-difference scheme in combination with the quasilinearization technique. Since the detailed description of the method is given elsewere[11,12], it is not presented here. The effects of step sizes $\Delta\eta$ and Δt^* and the edge of the boundary layer η_∞ on the solution have been studied and optimum values of $\Delta\eta$, Δt^* and η_∞ have been obtained. Consequently, we have taken $\Delta\eta = 0.05$ and $\Delta t^* = 0.1$ for computation. Also we have taken the values of the edge of the boundary layer (η_∞) between 4 and 8 depending on the values of the parameters. The results presented here are independent of step sizes and η_∞ at least up to 4th decimal place. For computation, the free stream velocity distributions have been taken to be of the form $\varphi(t^*) = 1 \pm \epsilon\, t^{*2}$ and $\varphi(t) = (1 + \epsilon_1 \cos\omega^* t^*)/(1 + \epsilon_1)$, where ϵ aed ϵ_1 are constants and ω^* is the frequency parameter. A typical case takes 15.2 seconds CPU time on DEC-1090 computer. For the self-similar case where $\varphi(t^*) = (1 - \lambda t^*)^{-1}$ a typical case takes 1.7 seconds CPU time.

4. Results and Discussion

Computations have been carried out for various values of the parameters M, b, j and λ. However, the results are presented only for some representative values of these parameters. Figs. 1 and 2 present the comparison with the results of the previous investigators. The results corresponding to the accelerating free stream velocity ($\varphi(t^*) = 1 + \epsilon\, t^{*2}$, $\epsilon > 0$) are presented in Figs. 3-6. The results corresponding to the fluctuating free stream velocity ($\varphi(t^*) = [1 + \epsilon_1 \cos\omega^* t^*]/(1 + \epsilon_1)$) are given in Fig. 7. The results for the self-similar case ($\varphi(t^*) = (1 - \lambda t^*)^{-1}$) are shown in Figs. 8–11.

In order to assess the accuracy of our method, we have compared our skin-friction and heat-transfer results $\left(f_w'', -g_w'\right)$ for both accelerating flow ($\varphi(t^*) = 1 + t^*$) and fluctuating flow ($\varphi(t^*) = 1 + \epsilon_1 \sin^2(\omega^* t^*)$ for $M = b = j = 0$ with the corresponding reults of Kumari and Nath[13] and found them in excellent agreement. However,

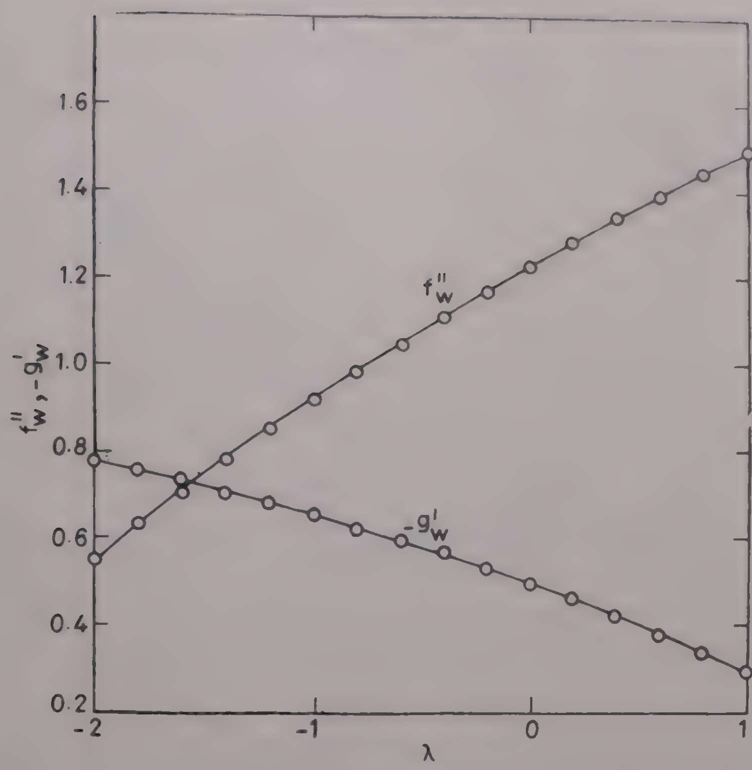

FIG. 2. Comparison of skin-friction and heat-transfer results $\left(f''_w, -g'_w\right)$ for $\varphi(t^*) = (1-\lambda t^*)^{-1}$ (self-similar case), $M = b = j = 0$, $Pr = 0.7$, ________, Present results; 0, Yang.

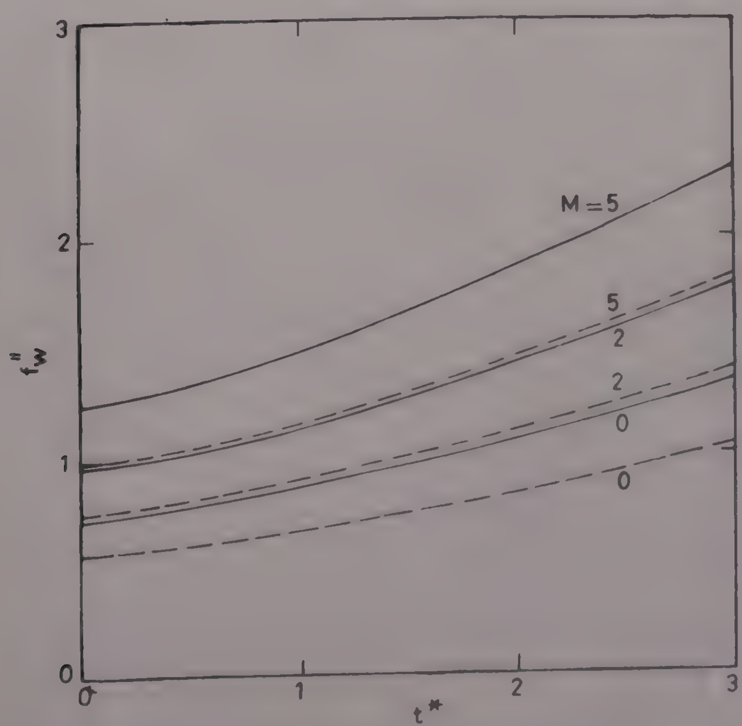

FIG. 3. Skin friction parameter $\left(f''_w\right)$ for $\varphi(t^*) = 1 + \epsilon t^{*2}$, $\epsilon = 0.25$, $b = 0.5$. ________, $j = 0$; ………, $j = 1$.

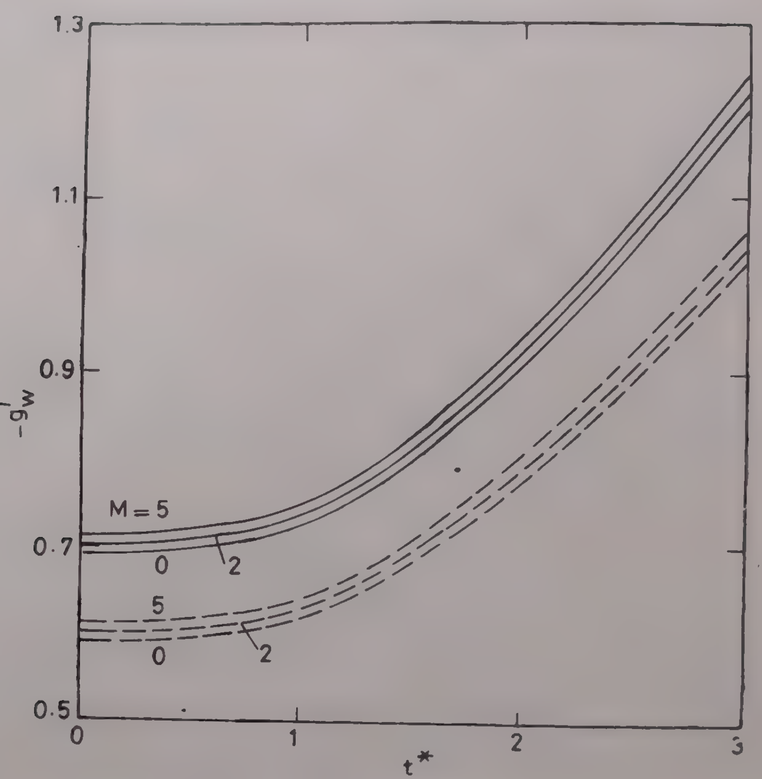

FIG. 4. Heat transfer parameter $\left(-g'_w\right)$ for $\varphi\ (t^*) = 1 + \epsilon t^{*2}$, $\epsilon = 0.25$, $b = 0.5$, $Pr = 0.73$ ———, $j = 0$; ………, $j = 1$.

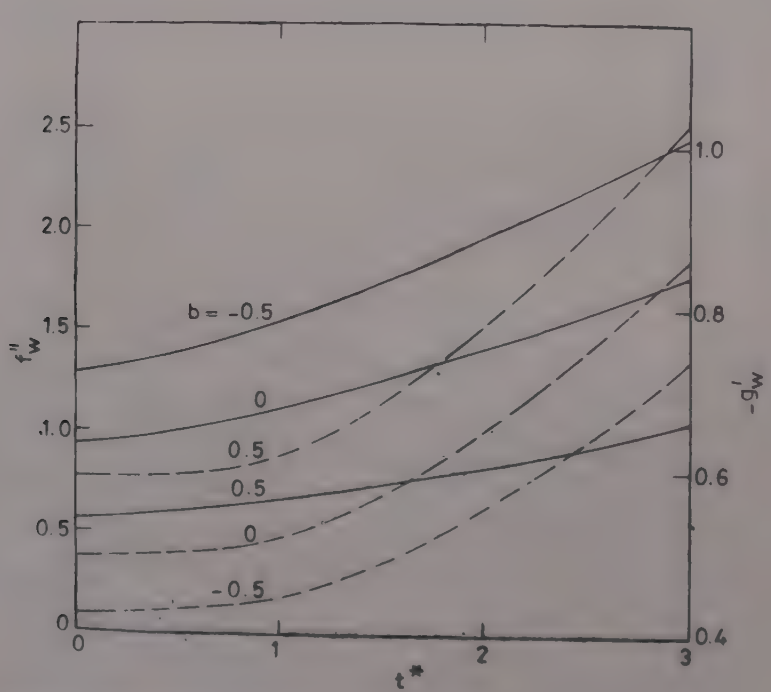

FIG. 5. Skin-friction and heat-transfer parameters $\left(f''_w, -g'_w\right)$ for $\varphi(t^*) = 1 + \epsilon\, t^{*2}, \epsilon = 0.25$, $M = 0$, $j = 1$, $Pr = 0.73$. ———, f''_w ; ……, $-g'_w$.

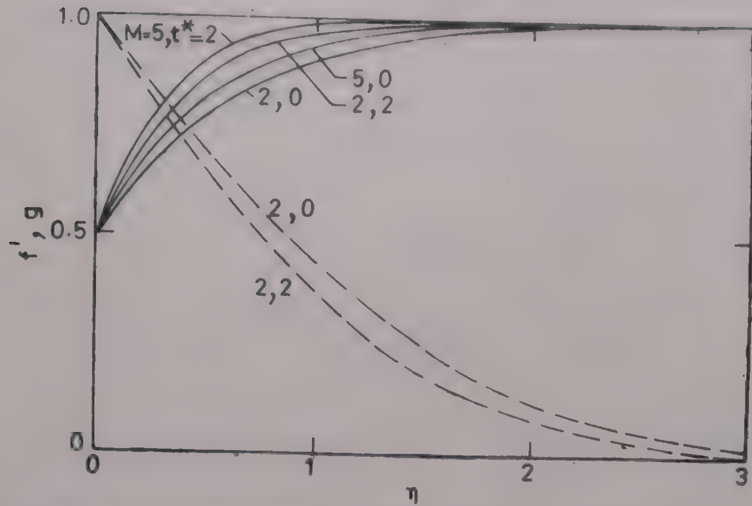

FIG. 6. Velocity and temperature profiles f', g) for $\varphi(t^*) = 1 + \epsilon t^{*2}$, $\epsilon = 0{,}25$, $b = 0.5$. $j = 1$, $Pr = 0.73$, ______, f'; ......, g.

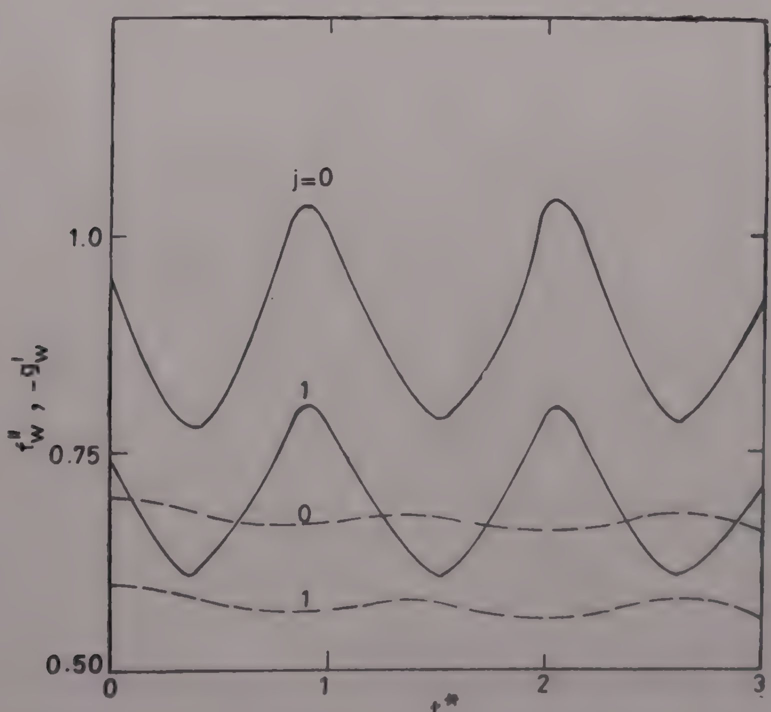

FIG. 7. Skin-friction and heat-transfer parameter $\left(f''_w, -g'_w\right)$ for $\varphi(t^*) = [1 + \epsilon_1 \cos (\omega^* t^*)]/(1+\epsilon_1)$, $\epsilon_1 = 0, 1$, $b = 0.5$, $M=2$, $\omega^* = 5.6$, $Pr = 0.73$. ______, f''_w; ..., $-g'_w$.

for the sake of brevity, the comparison is given only for $\varphi(t^*) = 1 + t^*$ in Fig. 1. We have also compared our skin-friction and heat transfer results $\left(f''_w, -g'_w\right)$ for the

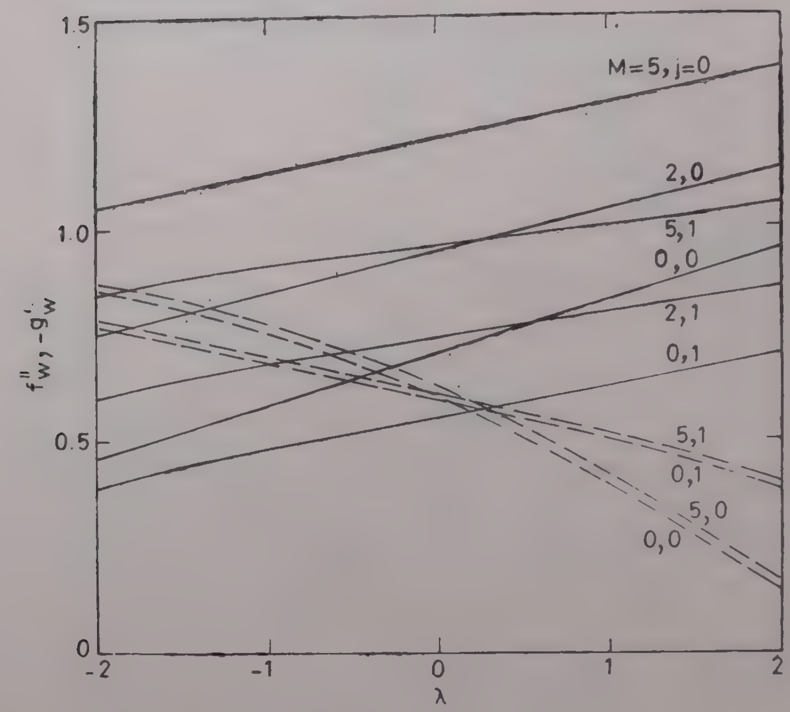

FIG. 8. Skin-friction and heat-transfer parameter $\left(f''_p, -g'_w\right)$ for $\varphi(t^*) = (1 - \lambda t^*)^{-1}$, b — 0.5 $j = 1$, $Pr = 0.73$. ———, f''_w; ······, $-g'_w$.

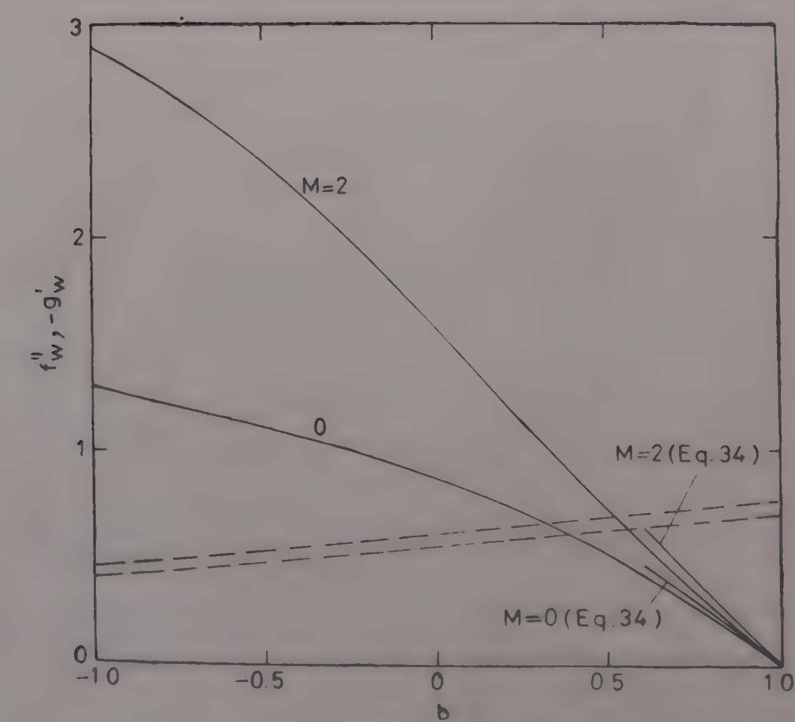

FIG. 9. Skin-friction and heat-transfer parameters $\left(f''_w, -g'_w\right)$ for $\varphi(t^*) = (1 - \lambda t^*)^{-1}$ $\lambda = -0.5$, $j = 1$, $Pr = 0.75$. ———, f''_w; ······ g'_w.

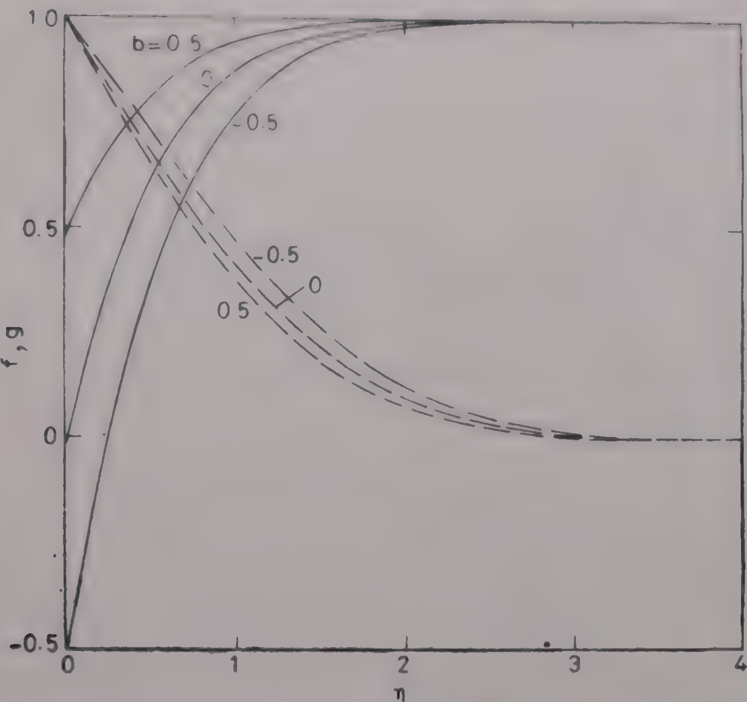

FIG. 10. Velocity and temperature profiles (f', g) for $\varphi(t^*) = (1 - \lambda t^*)^{-1}$, $M = 2$, $\lambda = -0.5$, $j = 0$, $Pr = 0.73$. ———, f'; ……, g.

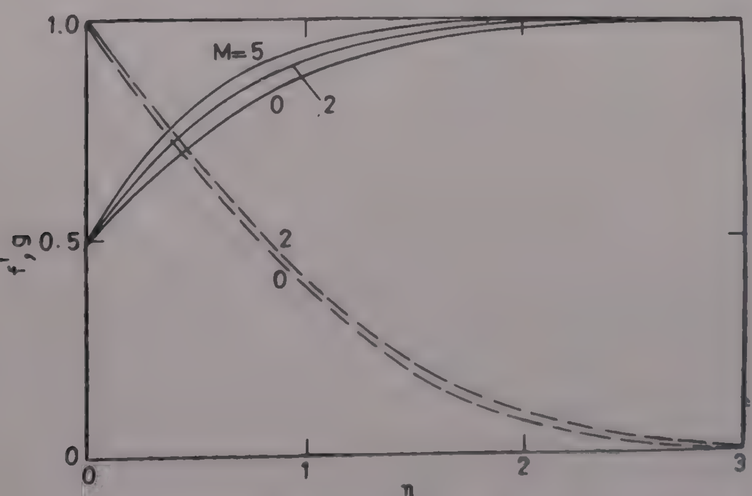

FIG. 11. Velocity and temperature profiles (f', g) for $\varphi(t^*) = (1 - \lambda t^*)^{-1}$, $\lambda = -0.5$, $j = 1$, $b = 0.5$, $Pr = 0.73$. ———, f'; ……, g.

self-similar flow for $M = b = j = 0$ with those of Yang[9] and they were also found to be in excellent agreement. The comparison is shown in Fig. 2.

The effect of the magnetic parameter M and time t^* on the skin friction and heat-transfer parameters $\left(f''_w, -g'_w\right)$ is shown in Figs. 3–4. For a given time t^*, the skin-friction and heat-transfer parameter $\left(f''_w, -g'_w\right)$ increase as M increases. Howevers,

the effect of M is more pronounced on the skin friction $\left(f''_w\right)$ than on the heat transfer $\left(-g'_w\right)$ because the magnetic term explicitly occurs only in the momentum equation. Similarly, for a given M, both f''_w and $-g'_w$ increase with time t^*, but the effect is more pronounced for $t^* > 1$. If the free stream velocity distribution is taken to be decelerating with time ($\varphi(t^*) = 1 - \epsilon t^{*2}, \epsilon > 0$, then f''_w and $-g'_w$ decrease as t^* increases. However, for the sake of brevity, the results for $\varphi(t^*) = 1 - \epsilon t^{*2}$, $\epsilon > 0$ are not presented here.

The effect of wall velocity b on the skin friction and heat transfer $\left(f''_w, -g'_w\right)$ is presented in Fig. 5. The skin friction is found to decrease as b increases, but the heat transfer increases. This is true for all values of M and t^*. This is due to the fact that as $b \to 1$ $(u_w \to u_e)$, the fluid tends to be inviscid. This causes considerable reduction in the skin friction as $b \to 1$. On the other hand, the difference between the wall temperature and free stream temperature $(T_w \to T_\infty)$ increases which results in increase in heat transfer.

The effect of the magnetic parameter M and time t^* on the velocity and temparature profiles $(f,'g)$ is shown in Fig. 6. The velocity and temparature profiles (f', g) become more steep as M or t^* increases, because of the reduction in the momentum and thermal boundary layer thicknesses. Also for a given M or t^*, the thermal boundary layer thickness, is more than the momentum boundary layer thickness.

The skin-friction and heat-transfer parameters $\left(f''_w, -g'_w\right)$ for the oscillatory free stream velocity given by $\varphi(t^*) = (1 + \epsilon_1 \cos(\omega^* t^*)/(1+\epsilon_1)$ are given in Fig. 7. It is observed that the skin friction $\left(f''_w\right)$ responds more to the fluctuations of the free stream as compared to the heat transfer $\left(-g'_w\right)$, because the skin friction parameter is directly proportional to the velocity gradient which is influenced more by the free stream velocity as compared to the temperature gradient.

As mentioned earlier, the results for the self-similar flow have been presented in Figs. 8–11, The effects of the magnetic parameter M and the unsteady parameter λ on the skin friction and heat transfer parameter $\left(f''_w, -g'_w\right)$ are shown in Fig. 8. The skin friction parameter $\left(f''_w\right)$ increases as the magnetic parameter M or unsteady parameter λ increases. However the heat transfer parameter $\left(-g'_w\right)$ increases as M increases, but it decreases as λ increases. The reason for such a behaviour is that both momentum and thermal boundary layer thicknesses decrease as M increases,

but as λ increases momentum boundary layer thickness decreases and thermal boundary layer thickness increases.

The effect of the velocity of the moving wall b on the skin friction and heat transfer parameters $\left(f_w'', -g_w'\right)$ is shown in Fig. 9. It also contains the skin frction results obtained analytically for $b \approx 1$ (see eqn. (34)). The analytical result is found to be in good agreement with the numerical result. The effect of b is found to be more pronounced on the skin friction parameter as compared to the heat-transfer parameter, $\left(-g_w'\right)$. It is observed that f_w'' decreases rapidly as b approaches 1 and $f_w'' = 0$ for $b = 1$. This is due to the fact that for $b = 1$, the flow becomes potential and hence the skin friction parameter f_w'' vanishes.

The effects of the wall velocity b and the magnetic parameter M on the velocity and temperature profiless $(f', -g)$ are given in Figs. 10 and 11, respectively. For a given M, the velocity profile f' becomes more steep as b decreases, but the temperature profile g becomes less steep. Similarly, for a given b, the velocity and temperature profiles (f', g) become more steep as the magnetic parameter M increases. The reason for such a behaviour has been explained earlier.

5. Conclusions

The skin friction and heat transfer results are found to be significantly affected by the free stream velocity, magnetic field and the wall velocity. However, their effects on the heat transfer is comparatively less as compared to that on the skin friction. The self-similar solution exists when the free stream velocity, wall velocity and the square of the magnetic field vary inversely as a linear function of time. The skin friction and heat transfer increase as the magnetic parameter increases. However, the skin friction decreases as the wall velocity increases, but the heat transfer increases. The skin friction and heat transfer for the axisymmetric flow are found to be less than those of the two-dimensional flow.

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CONVECTION IN A STRATIFIED FLOW IN AN INCLINED POROUS CHANNEL

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The flow of a fluid of variable viscosity in an inclined channel bounded by two permeable layers is considered. The flow between the permeable layers is governed by Navier-Stokes equations. The flow in the porous medium is governed by Darcy law. The velocity field and the temperature distribution are obtained and the results are discussed.

1. NOMENCLATURE

$\bar{q}$ = (u, v, w) velocity of the fluid

ρ, ρ_0 = density of the fluid, density at $T = T_0$

p = pressure

T = temperature

T_0 = ambient temperature

k = permeability of the porous medium

Gr = Grashof number

g = acceleration due to gravity

h = distance between porous beds

Q = Darcy velocity

Q_1 = average Darcy velocity in the upper bed

Q_2 = average Darcy velocity in the lower bed

ϕ = angle of the inclination of the channel to the horizontal

$\sigma = \dfrac{h}{\sqrt{k}}$ = dimensionless parameter

$\gamma = \beta h$ = dimensionless parameter

μ = coefficient of viscosity

ν = coefficient of kinematic viscosity

β = volumetric expansion coefficient

$T_0 - \frac{\Delta T}{2}$ = temperature of the upper bed

$T_0 + \frac{\Delta T}{2}$ = temperature of the lower bed.

2. Introduction

Convective flow problems involving porous media have important applications in various disciplines like petroleum engineering, geophysics, agricultural engineering and so on. The study of blood flow in pulmonary alvelor sheet where the fluid is between two permeable layers is of considerable importance in biomechanics.

Flow in a porous medium is assumed to be governed by either Darcy law or a non-Darcy law. Beavers and Joseph[2] postulated a slip boundary condition at the nominal surface of the porous bed. Further they discussed the Poiseuille flow over a permeable bed and verified BJ condition experimentally. Rudraiah[4] made extensive studies on problems of flows through/past porous media. Rudraiah and Wilfred[4] investigated the natural convection in an inclined channel bounded by porous media. Channabasappa and Ranganna[3] examined the flow of a fluid of variable viscosity over a permeable bed.

In this paper, an exact solution for the flow of a fluid of variable viscosity between two inclined porous beds is obtained. The flow is governed by Navier-Stokes equations in the free flow region and is assumed to be described by Darcy law in the lower and upper porous beds. The temperature distribution, the velocity distribution, the mass flow rate and its fractional increase are obtained and discussed.

3. Mathematical Formulation

Consider the flow of stratified fluid of variable viscosity between two inclined porous beds. The flow is possible because of the imbalance between pressure and buoyancy forces when the Grashof number is not equal to zero, Batchelor[1]. The channel is inclinedat an angle ϕ with the horizontal. The lower and upper permeable beds are kept at $T_0 + \frac{1}{2}\Delta T$ and $T_0 - \frac{1}{2}\Delta T$ temperatures respectively. The flow is in x-direction and z-direction is taken perpendicular to the beds. Cartesian coordinate system is used.

The following assumptions are made to derive the basic equations of motion :

(1) The flow is steady and Grashof number is small.

(2) The velocity component u in x-direction is a function of z and ϕ only.

(3) The dissipation function Φ is neglected, the heat transfer takes place by conduction only and therefore the temperature is linear in z.

(4) The pressure is a function of x and z only. $p = p(x, z)$, $\frac{\partial p}{\partial x} = \lambda$, a constant.

(5) The viscosity coefficient and density decay exponentially with z. i.e.,

$$\mu = \mu_0 e^{-\beta z}, \rho = \rho_0 e^{-\beta z}.$$

The basic equations for the flow under consideration are

$$\rho = \rho_0 [1 - \beta (T - T_0)] \qquad ...(3.1)$$

$$\frac{d^2u}{dz^2} - \beta \frac{du}{dz} = \frac{1}{\mu_0} (\lambda + \rho g \sin \phi) e^{\beta z} \qquad ...(3.2)$$

$$\frac{\partial p}{\partial z} + \rho g \cos \phi = 0 \qquad ...(3.3)$$

$$\frac{\partial u}{\partial x} = 0 \qquad ...(3.4)$$

$$\frac{d^2 T}{dz^2} = 0. \qquad ...(3.5)$$

The Darcy velocity is given by

$$Q = Q_0 e^{\beta z} \qquad ...(3.6)$$

where

$$Q_0 = - \frac{k}{\mu_0} (\lambda + \rho g \sin \phi).$$

The boundary conditions are

$$\frac{du}{dz} = - \frac{\alpha}{\sqrt{k}} (u_{B1} - Q_1) \text{ at } z = h/2 \qquad ...(3.7)$$

$$\frac{du}{dz} = \frac{\alpha}{\sqrt{k}} (u_{B2} - Q_2) \text{ at } z = - h/2 \qquad ...(3.8)$$

$$u = u_{B1} \text{ at } z = h/2 \qquad ...(3.9)$$

$$u = u_{B2} \text{ at } z = - h/2 \qquad ...(3.10)$$

$$T = T_0 - \tfrac{1}{2} \Delta T \text{ at } z = h/2 \qquad ...(3.11)$$

$$T = T_0 + \tfrac{1}{2} \Delta T \text{ at } z = - h/2 \qquad ...(3.12)$$

$$p = 0 \text{ at } x = 0, z = 0. \qquad ...(3.13)$$

4. Non-dimensionalization of the flow Quantities

The following non-dimensional parameters are introduced to make the governing equations and boundary conditions dimensionless.

$$u^* = \frac{u}{\nu/h},\ z^* = \frac{z}{h},\ x^* = \frac{x}{h},\ T^* = \frac{T}{\Delta T},\ T_0^* = \frac{T_0}{\Delta T}$$

$$p^* = \frac{p}{g\rho_0 h},\ u_{B1}^* = \frac{u_{B1}}{\nu/h},\ u_{B2}^* = \frac{u_{B2}}{\nu/h},\ Q^* = \frac{Q}{\nu/h},\ Q_1^* = \frac{Q_1}{\nu/h}$$

$$Q_2^* = \frac{Q_2}{\nu/h},\ \lambda^* = \lambda/g\,\rho_0,\ M^* = \frac{M}{\mu_0}. \qquad \ldots(4.1)$$

In view of (4.1), eqns. (3.1) - (3.13), after neglecting asterisks, become

$$\rho = \rho_0\,[1 - \beta\,(T - T_0)\,\Delta T] \qquad \ldots(4.2)$$

$$\frac{d^2u}{dz^2} - \gamma\,\frac{du}{dz} - \eta\left(\frac{\partial p}{\partial x} + \sin\phi\right)e^{\gamma z} + Gr\,(T - T_0)\,e^{\gamma z}\sin\phi = 0 \qquad \ldots(4.3)$$

$$\frac{\partial p}{\partial z} + \left[1 - \frac{Gr}{\eta}\,(T - T_0)\right]\cos\phi = 0 \qquad \ldots(4.4)$$

where

$$\eta = \frac{gh^3}{\nu^2},\ Gr = \eta\beta\,\Delta T$$

$$\frac{\partial u}{\partial x} = 0 \qquad \ldots(4.5)$$

$$\frac{d^2T}{dz^2} = 0 \qquad \ldots(4.6)$$

$$Q = -\,\frac{e^{\gamma z}}{\sigma^2}\,(-p + z\,Gr\sin\phi) \qquad \ldots(4.7)$$

where

$$P = -\eta\,(\lambda + \sin\phi).$$

$$\frac{du}{dz} = -\,\alpha\sigma\,(u_{B_1} - Q_1)\ \text{at}\ z = \tfrac{1}{2} \qquad \ldots(4.8)$$

$$\frac{du}{dz} = \alpha\sigma\,(u_{B_2} - Q_2)\ \text{at}\ z = -\tfrac{1}{2} \qquad \ldots(4.9)$$

$$u = u_{B_1}\ \text{at}\ z = \tfrac{1}{2} \qquad \ldots(4.10)$$

$$u = u_{B_2}\ \text{at}\ z = -\tfrac{1}{2} \qquad \ldots(4.11)$$

$$T = T_0 - \tfrac{1}{2}\ \text{at}\ z = \tfrac{1}{2} \qquad \ldots(4.12)$$

$$T = T_0 + \tfrac{1}{2} \text{ at } z = -\tfrac{1}{2} \qquad \ldots(4.13)$$

$$p = 0 \text{ at } x = 0,\ z = 0. \qquad \ldots(4.14)$$

5. Solution of Problem

The temperature distribution is obtained as given below on solving (4.6) and using the boundary conditions (4.12) and (4.13).

$$T - T_0 = -z. \qquad \ldots(5.1)$$

On solving (4.3), subject to the boundary conditions (4.8)-(4.11), the velocity field is obtained as

$$u = A + B\,e^{\gamma z} - \frac{Pz}{\gamma}\,e^{\gamma z} + \frac{e^{\gamma z}\,Gr \sin\phi}{\gamma}\left(\frac{z^2}{2} - \frac{z}{\gamma}\right). \qquad \ldots(5.2)$$

The constants A, B etc. are given by

$$a_1 A + a_2 B = a_3$$

$$b_1 A + b_2 B = b_3$$

$$a_1 = -\alpha\sigma$$

$$a_2 = -(\alpha\sigma + \gamma)\,e^{\gamma/2}$$

$$a_3 = \alpha\sigma\left[-\frac{P}{e\gamma}\,e^{\gamma/2} + \frac{e^{\gamma/2}\,Gr\sin\phi}{\gamma}\left(\frac{1}{8} - \frac{1}{2\gamma}\right) - Q_1\right]$$
$$+\ e^{\gamma/2}\left[-\frac{P}{2} - \frac{P}{\gamma} + \frac{Gr\sin\phi}{8} - \frac{Gr\sin\phi}{\gamma^2}\right]$$

$$b_1 = \alpha\sigma$$

$$b_2 = (\alpha\sigma - \gamma)\,e^{-\gamma/2}$$

$$b_3 = -\alpha\sigma\left[\frac{P}{2\gamma}\,e^{-\gamma/2} + \frac{e^{\gamma/2}\,Gr\sin\phi}{\gamma}\left(\frac{1}{8} + \frac{1}{2\gamma}\right) - Q_2\right]$$
$$+\ e^{-\gamma/2}\left[\frac{P}{2} - \frac{P}{\gamma} + \frac{Gr\sin\phi}{8} - \frac{Gr\sin\phi}{\gamma^2}\right]$$

$$A = \frac{a_3 b_2 - a_2 b_3}{a_1 b_2 - a_2 b_1}$$

$$B = \frac{a_1 b_3 - a_3 b_1}{a_1 b_2 - a_2 b_1}.$$

The slip velocities at the upper and lower beds are given by

$$u_{B_1} = A + Be^{\gamma/2} - \frac{P}{2\gamma}\,e^{\gamma/2} + \frac{e^{\gamma/2}\,Gr\sin\phi}{\gamma}\left(\frac{1}{8} - \frac{1}{2\gamma}\right) \qquad \ldots(5.3)$$

$$u_{B_2} = A + Be^{-\gamma/2} + \frac{P}{2\gamma} e^{-\gamma/2} + \frac{e^{-\gamma/2}\, Gr \sin\phi}{\gamma}\left(\frac{1}{8} + \frac{1}{2\gamma}\right). \quad \ldots(5.4)$$

The pressure distribution is given by

$$P = -\left[z + \frac{z^2\, Gr}{2\eta}\right]\cos\phi - \lambda x. \quad \ldots(5.5)$$

The average Darcy velocities in the upper and lower beds are given as

$$Q_1 = -\frac{e^{\gamma/2}}{\sigma^2}\left(-P + \tfrac{1}{2}\, Gr \sin\phi\right) \quad \ldots(5.6)$$

$$Q_2 = -\frac{e^{-\gamma/2}}{\sigma^2}\left(-P - \tfrac{1}{2}\, Gr \sin\phi\right). \quad \ldots(5.7)$$

6. Discussions

1. *Mass flow rate*—The dimensionless mass flow rate per unit width of the channel bounded by two inclined permeable beds is given by

$$M = \int_{1/2}^{1/2} e^{-\gamma z}\, u\, dz$$

$$= \frac{2A}{\gamma}\sinh\frac{\gamma}{2} + B + \frac{Gr \sin\phi}{24\gamma}. \quad \ldots(6.1)$$

Letting $\sigma \to \infty$ in (6.1), we obtain the mass flow rate per unit width of the channel bounded by rigid walls. It is given by

$$M_1 = \frac{2A_1}{\gamma}\sinh\frac{\gamma}{2} + B_1 + \frac{Gr \sin\phi}{24\,\gamma} \quad \ldots(6.2)$$

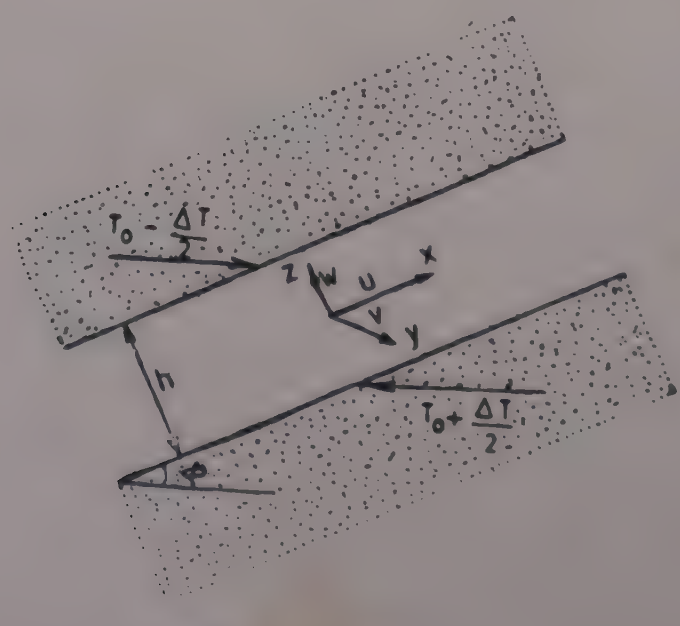

FIG. 1. Physical Model.

where

$$A_1 = -\left[\frac{P}{\gamma} + \frac{Gr \sin \phi}{\gamma^2}\right] \Big/ 2 \sinh \tfrac{1}{2}\gamma$$

$$B_1 = \coth \frac{\gamma}{2} \left[\frac{P}{2\gamma} + \frac{Gr \sin \phi}{4\gamma^2}\right] - \frac{Gr \sin \phi}{8\gamma}.$$

2. *Fractional increase*—The fractional increase in mass flow rate through the inclined porous channel over the inclined flow with rigid walls is

$$F = \frac{M - M_1}{M_1}$$

$$= \frac{48 \sinh \frac{1}{2} \gamma (A - A_1) + 24\gamma (B - B_1)}{48A_1 \sinh \frac{1}{2}\gamma + 24B_1 \gamma + Gr \sin \phi}. \qquad \ldots(6.3)$$

3. The velocity distribution (5.2) and fractional increase (6.3) are numerically computed for different values of P and σ with $Gr \sin \Phi = 25$, $\alpha = 0.1$ and $\gamma = 1$. They are shown in Fig. 2 and 3. The velocity profiles are in the

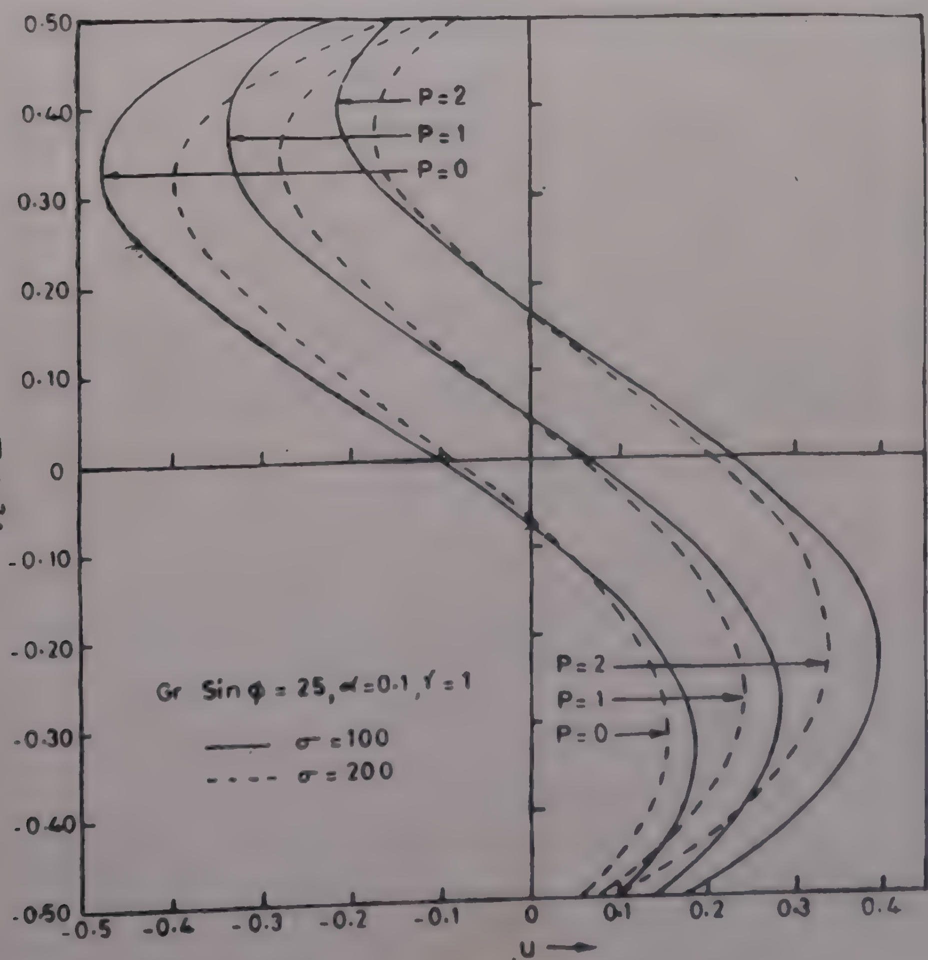

FIG. 2. Velocity profiles.

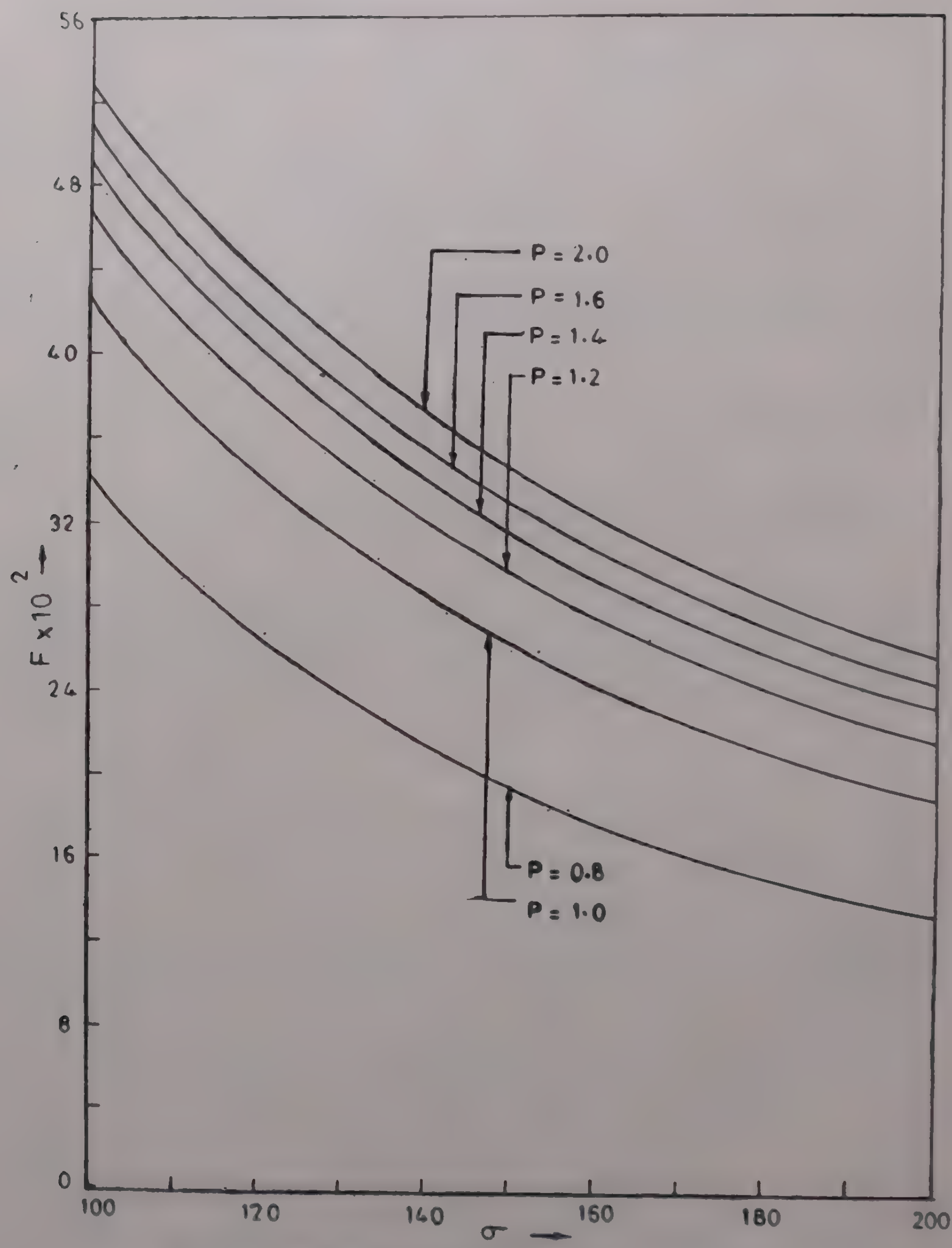

FIG. 3. Fractional increase in mass flow rate.

shape of S and resemle the velocity profiles of Rudraiah and Wilfred[5] for a fixed γ. For fixed σ, the velocity increases with the increase in P. For fixed P, the velocity increases in the bottom regions of the channel, with the increase in σ whereas the velocity decreases in the top regions of the channel with the increase in σ. The back flow is found in the upper half of the channel. It is due to the adverse pressure gradient P surpassing the action of the viscous forces in the region. In Fig. 3 fractional increase F is plotted against the permeability parameter σ for various values of P. For fixed P, F decreases with the increases in σ. The fractional increase raises with the increase in P. As P increases, the gap between these F curves narrows down.

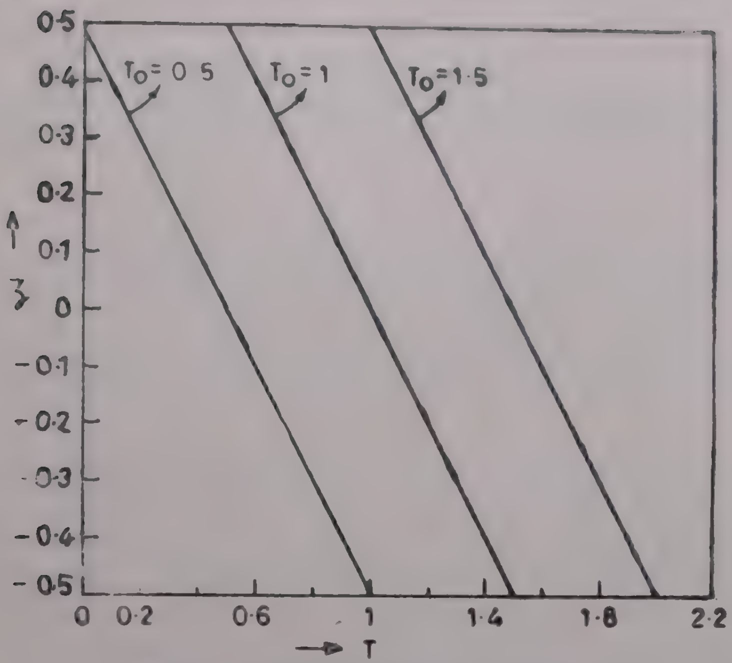

FIG. 4. Temperature Distribution.

The temperature is numerically evaluated for various values of T_0 and is shown in Fig. 4. The conclusions are given below.

Fixed parameter	Varying parameter	Behaviour of the variable	Effect on the temperature
T_0	z	increases	decreases
z	T_0	increases	increases

We note that the temperature is independent of the permeability k.

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TRANSIENT FREE CONVECTION FLOW AROUND TWO-DIMENSIONAL OR AXISYMMETRIC BODIES

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An analysis of unsteady free convection flow around two-dimensional or axisymmetric bodies heated impulsively is considered by taking the surface temperature as an exponential function of time. The method of matched asymptotic expansions is used to solve the Navier-Stokes equations including the buoyancy forces. This theory is applied for free convection flow past a horizontal circular cylinder and a sphere to bring out the importance of displacement and curvature effects on heat transfer and skin friction.

1. Introduction

When the free convection flow starts from rest, the unsteady terms in the momentum and energy equations are more significant than the inertia terms for a small time t_0

$$\frac{U_c\, t_0}{L} = \epsilon \ll 1 \qquad \text{.. (1)}$$

where L is the characteristic length, U_c is the characteristic velocity of the body. As $\epsilon \to 0$, $Gr \to \infty$ the free convection around a vertical flat plate is governed by the unsteady one dimensional flow equations. This has been considered for a step input in the surface temperature by Mohanty[1], Menond and Yang[2], Goldstein and Briggs[3]. Elliot[4] analysed the problem of unsteady free convection over two-dimensional or axisymmetric bodies for step input in the surface temperature. They[1-4] ignored the curvature and displacement effects. Gupta and Pop[5] considered the problem of Elliot and included the curvature effects. The effects of both curvature and displacement have been included in the analysis of Arunachalam and Rajappa[6], in which the surface temperature is a power function of time.

The aim of the present paper is to study the unsteady free convection flow past a two-dimensional or axisymmetric body heated impulsively and placed in an ambient fluid of temperature T_∞. The input in the surface temperature is taken as an exponential function of time which represents the phenomenon of the sudden heating of the body. The application of the present theory for an unsteady free convection flow past a horizontal circular cylinder and a sphere is carried out to bring out the importance of curvature and displacement effects on heat transfer and skin friction.

2. Statement of the Problem

Consider an incompressible, transient, nondissipative constant property laminar boundary layer flow induced by the buoyancy force $\rho \bar{g} \beta (T - T_{\infty})$ over a hot two-dimensional or axisymmetric body whose temperature on the surface is

$$T_w (\bar{t}) = T_{\infty} + \Delta t \, e^{(\bar{t}/t_0)} \qquad \ldots(2)$$

where T_{∞} is the undisturbed temperature of the ambient fluid which is at rest at infinity. The radius of curvature at the leading edge of the body is taken as characteristic length L, ΔT reference temperature and t_0 the investigation time of flow. We have the buoyancy induced velocity U_c given by

$$U_c = (g\beta\Delta \, TL)^{1/2}.$$

In terms of these quantities

$$Gr^{1/2} = \frac{U_c L}{\nu}, \; F = \frac{\alpha t.}{L^2}, \; \sigma = \frac{\nu}{\alpha} \text{ and } \epsilon = \sigma F_0 \sqrt{Gr} \qquad \ldots(3)$$

where ν is is the kinematic viscosity of the fluid, α the thermal diffusion coefficient. Gr, F_0 and σ are the Grashof, Fourier and Prandtl numbers respectively. Gr plays an important role in free convection and F_0 in unsteady flows.

In the case of unsteady viscous flows the parameter $\sqrt{\nu t_0}/L$ plays an important role. We take

$$\sqrt{\nu t_0}/L = \epsilon\sqrt{\gamma} \qquad \ldots(4)$$

where γ is a constant. When time t_0 is small $\sqrt{\nu t_0}/L$ is small and ϵ is small. γ denotes the ratio of $\sqrt{\nu t_0}/L$ to $\sigma F_0 \sqrt{Gr}$ which is considered to be a constant of $O(1)$.

Using curvilinear coordinate system shown in Fig. 1 the governing equations for unsteady free convection over a curved surface stated in nondimensional coordinates are

$$\frac{\partial}{\partial x} (h_2 h_3 u) + \frac{\partial}{\partial y} (h_1 h_3 v) = 0 \qquad \ldots(5)$$

$$\frac{\partial u}{\partial t} + \epsilon \left(\frac{u}{h_1} \frac{\partial u}{\partial x} + \frac{v}{h_2} \frac{\partial u}{\partial y} + \frac{kuv}{h_1 h_2} \right) + \frac{1}{h_1} \frac{\partial p}{\partial x}$$

$$= AH \sin \theta - \frac{\gamma \epsilon^2}{h_2 h_3} \frac{\partial}{\partial y} (h_3 \zeta_3) \qquad \ldots(6)$$

$$\frac{\partial v}{\partial t} + \epsilon \left(\frac{u}{h_1} \frac{\partial v}{\partial x} + \frac{v}{h_2} \frac{\partial v}{\partial y} - \frac{ku^2}{h_1 h_2} \right) + \frac{1}{h_2} \frac{\partial p}{\partial y}$$

$$= - AH \cos \theta + \frac{\gamma \epsilon^2}{h_1 h_3} \frac{\partial}{\partial x} (h_. \zeta_3) \qquad \ldots(7)$$

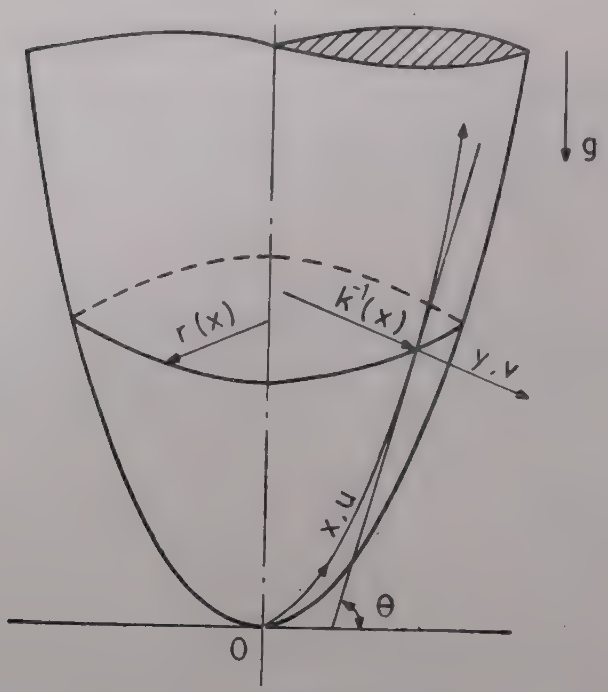

FIG. 1. Coordinate system.

$$\frac{\partial H}{\partial t} + \epsilon\left(\frac{u}{h_1}\frac{\partial H}{\partial x} + \frac{v}{h_2}\frac{\partial H}{\partial y}\right) = \frac{\gamma\epsilon^2}{\sigma h_1 h_3}\left[\frac{\partial}{\partial x}\left(\frac{h_3}{h_1}\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(h_1 h_3 \frac{\partial H}{\partial y}\right)\right] \quad \text{...(8)}$$

where

$$h_1 = 1 + ky,\ h_2 = 1,\ h_3 = (r + y \sin\theta)^j \quad \text{...(9)}$$

$$\zeta_3 = -\left(\frac{\partial u}{\partial y} + \frac{ku}{h_1} - \frac{1}{h_1}\frac{\partial v}{\partial x}\right) \quad \text{...(10)}$$

$$A = \frac{g\beta\,\Delta T\, t_0}{U_c} = \epsilon,\ p = \frac{\bar{p} - p_\infty}{\epsilon^{-1}\rho U_c^2},\ H = \frac{(T - T_\infty)}{\Delta T} \quad \text{...(11)}$$

k is the curvature at any point. k and θ are functions of x only. ζ_3 is the vorticity component. The buoyancy force is of the same order as the unsteady terms in the momentum equations. This implies that A is a constant of $O(\epsilon)$. This constant A has been taken as $O(1)$ in the article of Arunachalam and Rajappa[6]. In the present problem the Navier-Stokes equations including the energy equation are solved using the fact that A is of $O(\epsilon)$. The index $j = 0$ gives the two-dimensional flow and $j = 1$ axisymmetric flow.

Introducing the Stokes stream function ψ, such that

$$u = \frac{1}{h_3 h_3}\frac{\partial \psi}{\partial y},\ v = -\frac{1}{h_1 h_3}\frac{\partial \psi}{\partial x} \quad \text{...(12)}$$

the equation of continuity will automatically be satisfied. The boundary conditions to be satisfied are

$$t \leqq 0 : u = 0 = v,\ H = 0 \text{ for all } y$$

$$t > 0 : u = 0 = v,\ H = e^{t} \text{ at } y = 0 \qquad \text{...(13)}$$

$$u \to 0,\ H \to 0 \text{ as } y \to \infty.$$

Since the problem is a singular perturbation one where the small parameter ϵ is multiplying the highest derivative. One has to use the method of matched asymptotic expansions as given by Vandyke[7], to solve the problem. For the outer flow field we assume that

$$\psi = \Psi_1 + \epsilon\,\Psi_2 + \ldots, \qquad \text{...(14)}$$

and similar expansions in capital letters for u, v, H and p. A regular perturbation of equations (5) to (8) due to equations (14) shows that the outer flow field is always inviscid and irrotationl. Therefore from equations (10) and (12) we infer that Ψ_1, Ψ_2 ... are solutions of

$$\frac{\partial}{\partial y}\left(\frac{h_1}{h_2\,h_3}\,\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{h_2}{h_1\,h_3}\,\frac{\partial\psi}{\partial x}\right) = 0. \qquad \text{...(15)}$$

For inner flow field we introduce an innner variable z as

$$z = y/\epsilon \qquad \text{...(16)}$$

and expand the inner flow quantities as

$$u = u_1 + \epsilon\,u_2 + \ldots,\ p = p_1 + \epsilon p_2 + \ldots,\ H = H_1 + \epsilon H_2 + \ldots,$$

$$v = \epsilon v_1 + \epsilon^2\,v_2 + \ldots,\ \psi = \epsilon\psi_1 + \epsilon^3\,\psi_2 + \ldots, \qquad \text{...(17)}$$

Comparing the like powers of ϵ, we get from equations (5) to (8) the boundary layer equations to different orders :

First order equations are

$$\frac{\partial}{\partial x}\,(r^j\,u_1) + \frac{\partial}{\partial z}\,(r^j\,v_1) = 0 \qquad \text{...(18)}$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x} = \gamma\,\frac{\partial^2 u_1}{\partial z^2} \qquad \text{...(19)}$$

$$\frac{\partial p_1}{\partial z} = 0 \qquad \text{...(20)}$$

$$\frac{\partial H_1}{\partial t} = \frac{\gamma}{\sigma}\,\frac{\partial^2 H_1}{\partial z^2}. \qquad \text{...(21)}$$

Second order eqations are

$$\frac{\partial}{\partial x}(r^j u_2 + jz\, u_1 \sin\theta) + \frac{\partial}{\partial z}(r^j v_2 + kr^j\, v_1 z + jzv_1 \sin\theta) = 0 \qquad \ldots(22)$$

$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial z} + \frac{\partial p_2}{\partial x} - kz \frac{\partial p_1}{\partial x} = H_1 \sin\theta$$

$$+ \gamma \left[\frac{\partial^2 u_2}{\partial z^2} + \left(k + \frac{j}{r}\sin\theta\right)\frac{\partial u_1}{\partial z}\right] \qquad \ldots(23)$$

$$\frac{\partial p_2}{\partial z} = 0 \qquad \ldots(24)$$

$$\frac{\partial H_2}{\partial t} + u_1 \frac{\partial H_1}{\partial x} + v_1 \frac{\partial H_1}{\partial z} = \frac{\gamma}{\sigma}\left[\frac{\partial^2 H_2}{\partial z^2} + \left(k + \frac{j}{r}\sin\theta\right)\frac{\partial H_1}{\partial z}\right]. \qquad \ldots(25)$$

Third order equations are

$$\frac{\partial}{\partial x}(r^j u_3 + jz\, u_2 \sin\theta) + \frac{\partial}{\partial z}(r^j v_3 + jkv_1 z^2 \sin\theta + r^j zkv_2 + jzv_2 \sin\theta) = 0 \qquad \ldots(26)$$

$$\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_1}{\partial x} - kzu_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_2}{\partial z} + v_2 \frac{\partial u_1}{\partial z} + ku_1 v_1$$

$$+ \frac{\partial p_3}{\partial x} - kz \frac{\partial p_2}{\partial x} = H_2 \sin\theta + \gamma \left[\frac{\partial^2 u_3}{\partial z^2} + \left(k + \frac{j}{r}\sin\theta\right)\frac{\partial u_2}{\partial z}\right.$$

$$\left. - \left(k^2 + \frac{j}{r}\sin^2\theta\right) z \frac{\partial u_1}{\partial z} + k\frac{j}{r}\sin\theta\, u_1 - \frac{\partial^2 v_1}{\partial z\,\partial x}\right] \qquad \ldots(27)$$

$$\frac{\partial v_1}{\partial t} - ku_1^2 + \frac{\partial p_3}{\partial z} = -H_1 \cos\theta - \gamma \left[\frac{\partial^2 u_1}{\partial x\,\partial z} + \frac{j}{r}\frac{dr}{dx}\frac{\partial u_1}{\partial z}\right] \qquad \ldots(28)$$

$$\frac{\partial H_3}{\partial t} + u_1 \frac{\partial H_2}{\partial x} + u_2 \frac{\partial H_1}{\partial x} - kzu_1 \frac{\partial H_1}{\partial x} + v_1 \frac{\partial H_2}{\partial z} + v_2 \frac{\partial H_1}{\partial z}$$

$$= \frac{\gamma}{\sigma}\left[\frac{\partial^2 H_1}{\partial x^2} + \left(k + \frac{j}{r}\sin\theta\right)\frac{\partial H_2}{\partial z} - \left(\frac{j}{r}\sin^2\theta + k^2\right)\right.$$

$$\left.\frac{\partial H_1}{\partial z} + \frac{\partial^2 H_3}{\partial z^2} + jr\frac{dr}{dx}\frac{\partial H_1}{\partial x}\right]. \qquad \ldots(29)$$

The equations have to be solved subject to the boundary conditions in equation (13) and matching conditions which come from the outer flow solution.

3. Solution of the Problem

First Approximation

As the ambient fluid is at rest and temperature does not vary across the boundary layer at a large distance from the surface of the body, the first order outer flow quantities are

$$p_1 = 0,\ H_1 = 0,\ \Psi_1 = 0. \qquad \ldots(30)$$

As a consequence of equation (20) noting that $p_1 = p_1(x, z)$ and pressure does not vary across the boundary layer, we have

$$\frac{\partial p_1}{\partial x} = \left(\frac{\partial P_1}{\partial x}\right)_{y=0} = 0. \qquad \ldots(31)$$

On introducing a similarity variable η as

$$\eta = z\left(\frac{\sigma}{\gamma}\right)^{1/2} \qquad \ldots(32)$$

and seeking solutions H_1 and u_1 in the form

$$H_1(x, \eta, t) = e^t Q_1(\eta) \qquad \ldots(33)$$

$$u_1(x, \eta, t) = e^t f_1(\eta). \qquad \ldots(34)$$

Equations (21) and (19) reduces to the form

$$Q_1'' - Q_1 = 0 \qquad \ldots(35)$$

$$\sigma f_1'' - f_1 = 0 \qquad \ldots(36)$$

where primes denote differentiation with respect to η. The boundary and matching conditions are

$$Q_1(0) = 1, Q_1(\infty) = 0 \qquad \ldots(37)$$

$$f_1(0) = 0 = f_1(\infty). \qquad \ldots(38)$$

The solution of (35) subject to (37) is

$$Q_1(\eta) = e^{-\eta}. \qquad \ldots(39)$$

The solution of (36) subject to (38) is

$$f_1(\eta) = 0. \qquad \ldots(40)$$

On introducing the stream function

$$\psi_1 \text{ as } u = r^{-s}\frac{\partial \psi_1}{\partial z}, v = -r^{-s}\frac{\partial \psi_1}{\partial x}.$$

We have

$$v_1(x, \eta, t) = 0 \text{ and } \psi_1(x, \eta, t) = 0. \qquad \ldots(41)$$

The second order outer flow quantities are

$$p_2 = 0, H_2 = 0, \psi_2 = 0. \qquad \ldots(42)$$

Second order inner flow gives

$$\frac{\partial p_2}{\partial x} = \left(\frac{\partial P_2}{\partial x}\right)_{y=0} = 0. \qquad \ldots(43)$$

Equation (25) after simplification becomes

$$\frac{\partial H_2}{\partial t} - \frac{\gamma}{\sigma}\frac{\partial^2 H_2}{\partial z^2} = \left(\frac{\gamma}{\sigma}\right)^{1/2}\left(k + \frac{j}{r}\sin\theta\right) e^t \, Q_1'(\eta). \qquad \ldots(44)$$

Seeking a solution for H_2 in the form

$$H_2(x, \eta, t) = \left(\frac{\gamma}{\sigma}\right)^{1/2}\left(k + \frac{j}{r}\sin\theta\right) e^t \, Q_2(\eta).$$

Equation (44) becomes

$$Q_2'' - Q_2 = - Q_1' \qquad \ldots(45)$$

$$Q_2(0) = 0 = Q_2(\infty). \qquad \ldots(46)$$

The solution of (45) subject to (46) is

$$Q_2(\eta) = - \frac{\eta}{2} e^{-\eta}. \qquad \ldots(47)$$

Equation (23) after simplification takes the form

$$\frac{\partial u_2}{\partial t} - \gamma \frac{\partial^2 u_2}{\partial z^2} = e^t \sin\theta \, Q_1. \qquad \ldots(48)$$

Seeking a solution for u_2 in the form

$$u_2(x, \eta, t) = e^t \sin\theta \, f(\eta)$$

equation (48) becomes

$$\sigma f'' - f = - Q_1 \qquad \ldots(49)$$

$$f(0) = 0 = f(\infty). \qquad \ldots(50)$$

The solution of (49) subject to (50) is

$$f(\eta) = \frac{\eta}{2} e^{-\eta}, \text{ for } \sigma = 1$$

$$= \frac{1}{(\sigma - 1)} (e^{-\eta_1} - e^{-\eta}), \text{ for } \sigma \neq 1, \; \eta_1 = \eta/\sqrt{\sigma}. \qquad \ldots(51)$$

From eqn. (22), introducing the stream function ψ_2 as

$$u_2 = r^{-j}\frac{\partial \psi_2}{\partial z}, \; v_2 = - r^{-j}\frac{\partial \psi_2}{\partial x}$$

we have

$$v_2(x, \eta, t) = - \frac{1}{r^j}\left(\frac{\gamma}{\sigma}\right)^{1/2} e^t \frac{\partial}{\partial x}(r^j \sin\theta)\, \phi(\eta) \qquad \ldots(52)$$

$$\psi_2(x, \eta, t) = \left(\frac{\gamma}{\sigma}\right)^{1/2} e^t r^j \sin\theta\, \phi(\eta) \qquad \ldots(53)$$

where

$$\phi(\eta) = \int_0^\eta f(x)\, dx$$

$$= \tfrac{1}{2}[1 - e^{-\eta} - \eta e^{-\eta}], \text{ for } \sigma = 1 \qquad \ldots(54)$$

$$= \frac{1}{(\sigma - 1)}[\sqrt{\sigma} - \sqrt{\sigma}\, e^{-\eta_1} + e^{-\eta} - 1], \text{ for } \sigma \neq 1,\ \eta_1 = \eta/\sqrt{\sigma}.$$

Third order outer flow

Since v_2 does not vanish as $\eta \to \infty$, the second order inner solution induces a correction in the outer inviscid solution specified by the stream function $\psi_3(x, y, t)$. Since $\Psi_2(x, y, t)$ is zero, a matching of Ψ as $y \to 0$ with $\epsilon^2 \psi_2$ to the second order of ϵ as $\eta \to \infty$, gives

$$\Psi_3(x, 0, t) = e^t r^j \left(\frac{\gamma}{\sigma}\right)^{1/2} \sin\theta\, \phi(\infty) \qquad \ldots(55)$$

where

$$\phi(\infty) = \frac{1}{\sqrt{\sigma} + 1}. \qquad \ldots(56)$$

This signifies an induced outer flow by the boundary layer, Ψ_3 satisfies eqns. (15) and (55), and vanishes as $y \to \infty$. Hence solving for Ψ_3, and using eqn. (12), we can get $U_3(x, y, t)$ so that

$$U_3(x, 0, t) = e^t D(x) \qquad \ldots(57)$$

where $D(x)$ is a function of x. Matching the inner and outer expressions for the x-component of velocity and pressure gradient to $O(\epsilon^2)$, we get as $\eta \to \infty$,

$$u_3 = e^t D(x) \qquad \ldots(58)$$

$$\frac{\partial p_3}{\partial x} = -\, e^t D(x). \qquad \ldots(59)$$

Third order inner flow

Rewriting eqn. (27) in the form

$$\frac{\partial u_3}{\partial t} - \gamma \frac{\partial^2 u_3}{\partial z^2} = e^t S_1 \left[k(Q_1 + Q_2 + \sigma f') + \frac{j}{r}\sin\theta(Q_2 + \sigma f') + \frac{D(x)}{S_1}\right] \qquad \ldots(60)$$

where

$$S_1(x) = \left(\frac{\gamma}{\sigma}\right)^{1/2} \sin\theta. \qquad \ldots(61)$$

Seeking a solution for u_3 as

$$u_3(x, \eta, t) = e^t \left[S_1\, k F_k(\eta) + S_1 \frac{j}{r} \sin\theta\, F_j(\eta) + D(x)\, F_D(\eta) \right]$$

where

$$F_k(\eta) = \frac{\eta}{2} e^{-\eta} - \frac{\eta^2}{4} e^{-\eta}, \text{ for } \sigma = 1$$

$$= \frac{1}{(\sigma - 1)} \left[\frac{-\sigma^3 + 2\sigma^2 + \sigma - 1}{(\sigma - 1)} e^{-\eta_1} - e^{-\eta} - \frac{\sigma}{(\sigma - 1)} e^{-\eta} - \frac{\eta}{2} e^{-\eta_1} \right]$$

$$- \frac{e^{-\eta}}{2} (\eta + \eta\sigma - 2\sigma), \text{ for } \sigma \neq 1,\ \eta_1 = \frac{\eta}{\sqrt{\sigma}} \qquad \ldots(62)$$

$$F_j(\eta) = - \frac{\eta^2}{4} e^{-\eta}, \text{ for } \sigma = 1$$

$$= \frac{1}{(\sigma - 1)} \left[\frac{\sigma^2 (2 - \sigma)}{(\sigma - 1)} e^{-\eta_1} - \frac{\sigma}{(\sigma - 1)} e^{-\eta} - \frac{\eta}{2} e^{-\eta_1} \right.$$

$$- \frac{e^{-\eta}}{2} (\eta + \eta\sigma - 2\sigma), \text{ for } \sigma \neq 1,\ \eta_1 = \frac{\eta}{\sqrt{\sigma}} \qquad \ldots(63)$$

$$F_D(\eta) = 1 - e^{-\eta}, \text{ for } \sigma = 1$$

$$= 1 - e^{-\eta_1}, \text{ for } \sigma \neq 1,\ \eta_1 = \frac{\eta}{\sqrt{\sigma}}. \qquad \ldots(64)$$

These equations form a set of universal velocity profiles, where the suffix k denotes the longitudinal curvature effect, j denotes the transverse curvature effect and D denotes the displacement effect.

The energy equation (29) after possible simplifications takes the form

$$\frac{\partial H_3}{\partial t} - \frac{\gamma}{\sigma} \frac{\partial^2 H_3}{\partial z^2} = e^{2t} \left(k + \frac{j}{r} \sin\theta \right) \cos\theta \phi(\eta)\ Q_1' + e^t \left(\frac{\gamma}{\sigma} \right)^{1/2}$$

$$\left[k^2 (Q_2' - \eta Q_1') + \left(\frac{j}{r} \sin\theta \right)^2 Q_2' - j \left(\frac{\sin\theta}{r} \right)^2 \right.$$

$$\left. \eta\, Q_1' + 2k \frac{j}{r} \sin\theta\, Q_2' \right]. \qquad \ldots(65)$$

Seeking a solution for H_3 in the from

$$H_3(x, \eta, t) = e^{2t} \left(k + \frac{j}{r} \sin\theta \right) \cos\theta\, g_1(\eta) + e^t \left(\frac{\gamma}{\sigma} \right)^{1/2} \left[k^2 g_2(\eta) \right.$$

$$+ \left(\frac{j}{r} \sin\theta \right)^2 g_3(\eta) + j \left(\frac{\sin\theta}{r} \right)^2 g_4(\eta)$$

$$\left. + k \frac{j}{r} \sin\theta\, g_5(\eta) \right]$$

where

$$g_1(\eta) = \frac{5}{4} e^{-\sqrt{2}\eta} - \frac{e^{-\eta}}{2} - \frac{\eta}{4} e^{-2\eta} - \frac{e^{-2\eta}}{2}, \text{ for } \sigma = 1$$

$$= \frac{1}{(\sigma - 1)} \left[\frac{-(3 + 4\sqrt{\sigma} - 7\sigma)}{\sqrt{2}(1 + 2\sqrt{\sigma} - \sigma)} e^{-\sqrt{2}\eta} + \frac{e^{-2\eta}}{2} + (1 - \sqrt{\sigma}) e^{-\eta} \right.$$

$$\left. - \frac{\sigma\sqrt{\sigma}}{(1 + 2\sqrt{\sigma} - \sigma)} e^{-\left(1 + \frac{1}{\sqrt{\sigma}}\right)\eta} \right], \text{ for } \sigma \neq 1 \qquad \text{..(66)}$$

$$g_2(\eta) = \frac{\eta}{8} e^{-\eta}(1 + 3\eta) \qquad \text{...(67)}$$

$$g_3(\eta) = \frac{\eta}{8} e^{-\eta}(\eta - 1) \qquad \text{...(68)}$$

$$g_4(\eta) = \frac{\eta}{4} e^{-\eta}(\eta + 1) \qquad \text{...(69)}$$

$$g_5(\eta) = \frac{\eta}{4} e^{-\eta}(\eta - 1) \qquad \text{...(70)}$$

where g_1 is the convection effect on the curved surface, g_2, g_3, g_4 and g_5 are the effects proportional to the curvature.

4. Skin Friction and Heat Transfer

Skin friction coefficient C_f is given by the formula

$$C_f = \frac{\left(\mu \frac{\partial \bar{u}}{\partial \bar{y}}\right)_w}{\left(\frac{1}{2} \rho U_c^2\right)}.$$

We have

$$C_f (F_0 Gr)^{1/2} = 2\epsilon\, e^t \sin\theta \left\{ f'(0) + \left(\frac{\gamma}{\sigma}\right)^{1/2} \epsilon \left[kF_k'(0) + \frac{j}{r} \sin\theta\, F_j'(0) \right.\right.$$

$$\left.\left. + \frac{D(x)}{S_1} F_D'(0) \right] \right\} \qquad \text{...(71)}$$

where suffix w denotes the value of the expression at the wall. If q_w denotes the heat transfer rate at the surface and Nu, the Nusselt nubmer based on the characteristic length L of the flow, then

$$Nu = \frac{q_w L}{K(T_w - T_\infty)} \text{ where } q_w = -\left(K \frac{\partial T}{\partial \bar{y}}\right)_w .$$

Hence

$$Nu\ F_0^{\frac{1}{2}} = - \Big\{ Q_1'(0) + \epsilon \left(\frac{\gamma}{\sigma}\right)^{\frac{1}{2}} \left(k + \frac{j}{r}\sin\theta\right) Q_2'(0)$$

$$+ \epsilon^2 \Big[e' \left(k + \frac{j}{r}\sin\theta\right) \cos\theta\, g_1'(0)$$

$$+ \left(\frac{\gamma}{\sigma}\right)^{\frac{1}{2}} \Big[k^2 g_2'(0) + \left(\frac{j\sin\theta}{r}\right)^2 g_3'(0) + j\left(\frac{\sin\theta}{r}\right)^2 g_4'(0)$$

$$+ k\frac{j}{r}\sin\theta\, g_5'(0)\Big]\Big]\Big\} + O(\epsilon^3). \qquad ...(72)$$

5. Applications

The present theory is applied to the unsteady buoyancy induced two-dimensional flow around a horizontal circular cylinder and axisymmetric flow around a sphere. Taking radius as the characteristic length L, and introducing the polar coordinates (R, θ) to get the outer flow, we have in the case of cylinder

$$j = 0,\ k = 1,\ h_1 = 1 + y = R,\ h_2 = 1,\ h_3 = 1,\ x = \theta. \qquad ...(73)$$

So, eqn. (15) takes the form

$$\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{\partial}{\partial x}\left(\frac{1}{R}\frac{\partial\psi}{\partial\psi}\right) = 0. \qquad ...(74)$$

Since Ψ_3 satisfies eqns. (74) and (55) with $j = 0$ and $x = \theta$, and vanishes as $R \to \infty$, we have

$$\Psi_3(x, R, t) = e'\left(\frac{\gamma}{\sigma}\right)^{\frac{1}{2}} \sin x\, \phi(\infty)/R. \qquad ...(75)$$

Therefore

$$U_3(x, 0, t) = \left(\frac{\partial\Psi_3}{\partial R}\right)_{R\to 1} = -e'\left(\frac{\gamma}{\sigma}\right)^{\frac{1}{2}} \sin x\, \phi(\infty) \qquad ...(76)$$

and

$$D(x) = -\left(\frac{\gamma}{\sigma}\right)^{\frac{1}{2}} \sin x\, \phi(\infty). \qquad ...(77)$$

In the case of a sphere also we arrive at the same expression for $D(x)$ given in eqn. (77), where we have

$$j = 1,\ k = 1,\ h_1 = 1 + y = R,\ h_2 = 1,\ \theta = x,\ r = \sin x,$$
$$h_3 = (1 + y)\sin x = R\sin x. \qquad ...(78)$$

The average Nusselt number Nu based on diameter d of the cylinder is

$$\overline{Nu_c} = \frac{\bar{h}d}{K} = \frac{1}{2\pi}\,\frac{d}{K}\int_0^{2\pi} h\,dx = \frac{1}{2\pi}\int_0^{2\pi} Nu\,dx$$

$$= -\,F_0^{-1/2}\left\{-1 + \frac{\epsilon}{2\sqrt{\sigma}}\left[-1 + \frac{\epsilon}{4}\right]\right\}. \qquad \ldots(79)$$

A similar averaging over the surface of the sphere will lead to the result

$$\overline{Nu_S} = -\,F_0^{-1/2}\left[-1 - \frac{\epsilon}{\sqrt{\sigma}}\right]. \qquad \ldots(80)$$

6. Discussion

It is to be noted that the first order inner and outer solution of the velocity distribution are identically zero. This shows that in free convection flows the velocity distribution is of $O(\epsilon)$. The velocity and temperature distributions are plotted in Figs. 2 to 5 for a typical value of Prandtl number $\sigma = 1$. Figure 2 shows the second order velocity profiles f and ϕ which indicate respectively the variation of u_2 and v_2 across the boundary layer. u_2 reaches maximum when $\eta = 1.2$. Figure 3 shows the third order velocity profiles. F_D denotes the contribution to the velocity due to displacement on the boundary layer. Its effect is very large compared with the curvature effects. The curvature contributions F_k and F_j are of $O(10^{-1})$, whereas the contribution of F_D is of $O(1)$.

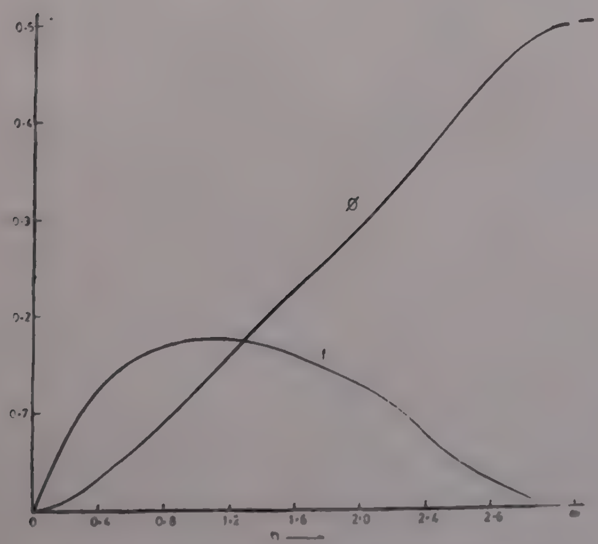

FIG. 2. Second order velocity profiles.

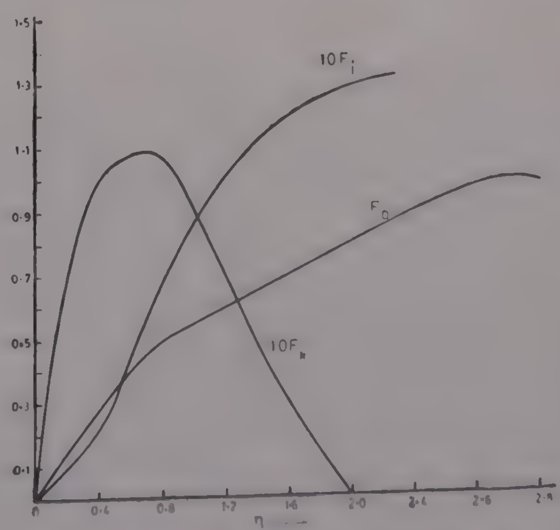

FIG. 3. Third order velocity profiles.

Figure 4 shows a plot of temperature profiles. Q_1 is the first order contribution which is due to transient conduction near the surface initially. Q_2 is the curvature effect. The terms proportional to k and j in the temperature distribution denote the longitudinal and transverse curvature effects. There is no effect due to the presence

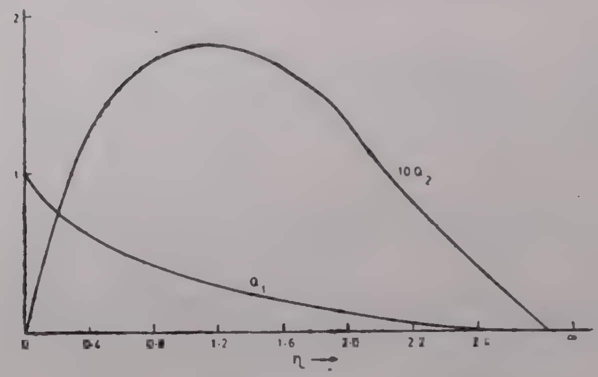

FIG. 4. First and second order temperature profiles.

of buoyancy force on the temperature even in the second order inner flow, since the first order convection velocities u_1 and v_1 vanish identically.

The temperature profiles of the third order solutions g_1, g_2, g_3, g_4 and and g_5 are shown in Fig. 5. g_1 is the effect due to convection on the curved surface which has

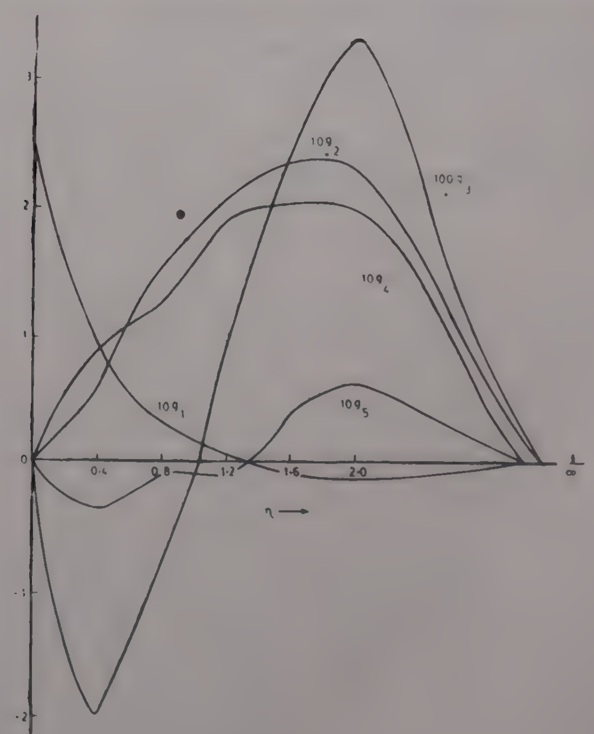

FIG. 5. Third order temperature profiles.

both longitudinal and transverse curvature. g_2, g_3, g_4 and g_5 are due to an interaction of the conduction terms and are independent of Prandtl number σ.

The variation of skin friction for a fixed value of $t = 0.5$ is shown in Fig. 6. From equation (71) it is to be noted that the terms (III order) proportional to k, j and

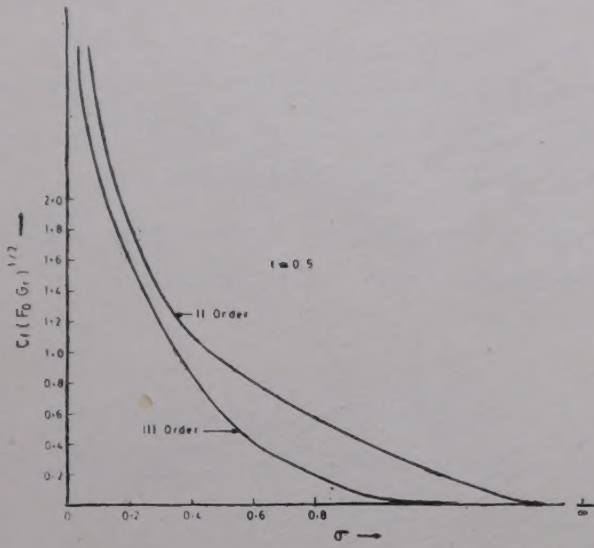

FIG. 6. $C_f\,(F_0\,G_r)^{1/2}$ versus σ in the case of cylinder.

D denote the contribution to skin friction due to longitudinal curvature, transverse curvature and displacement speed. These effect reduce the skin friction. As t increases the skin friction increases.

A plot of Nusselt number versus Prandtl number is shown in Fig. 7. First and second order terms are independent of time. When the Prandtl number is very large higher order terms tend to zero, so that the first order solution becomes the asymptotic solution for large σ. As t increases rate of heat transfer increases. There is an increase in heat transfer rate at the surface of the body due to curvature.

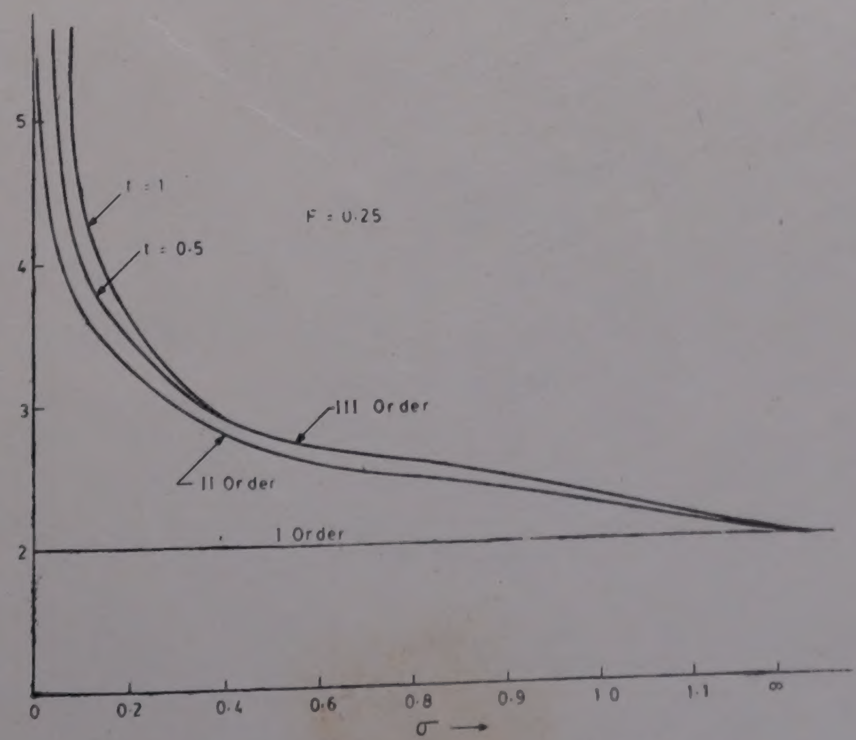

FIG. 7. Nu versus σ in the case of cylinder.

In the case of a circular cylinder and a sphere the displacement speed $D(x)$ given in eqn. (77) when substituted in eqn. (71) will give the skin friction coefficient C_f. From eqns. (79) and (80) it is to be noted that the average Nusselt number are functions of the Prandtl number σ. As $\sigma \to \infty$, the higher order effects vanish leaving the first order terms which is due to pure conduction.

The present analysis is applicable as in the case of cylinder and sphere to any two-dimensional or axisymmetric body whose curvatures and displacement speeds are known.

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